# ALGEBRA
## An Algorithmic Treatment

**Kenneth E. Iverson**

```
      X←1 2 3 4 5 6 7
      X×X
1 4 9 16 25 36 49

      X∘.×X
 1  2  3  4  5  6  7
 2  4  6  8 10 12 14
 3  6  9 12 15 18 21
 4  8 12 16 20 24 28
 5 10 15 20 25 30 35
 6 12 18 24 30 36 42
 7 14 21 28 35 42 49

      X∘.-X
 0 ¯1 ¯2 ¯3 ¯4 ¯5 ¯6
 1  0 ¯1 ¯2 ¯3 ¯4 ¯5
 2  1  0 ¯1 ¯2 ¯3 ¯4
 3  2  1  0 ¯1 ¯2 ¯3
 4  3  2  1  0 ¯1 ¯2
 5  4  3  2  1  0 ¯1
 6  5  4  3  2  1  0

      X∘.≥X
1 0 0 0 0 0 0
1 1 0 0 0 0 0
1 1 1 0 0 0 0
1 1 1 1 0 0 0
1 1 1 1 1 0 0
1 1 1 1 1 1 0
1 1 1 1 1 1 1

      N←X-4
      N
¯3 ¯2 ¯1 0 1 2 3

      N∘.×N
 9  6  3  0 ¯3 ¯6 ¯9
 6  4  2  0 ¯2 ¯4 ¯6
 3  2  1  0 ¯1 ¯2 ¯3
 0  0  0  0  0  0  0
¯3 ¯2 ¯1  0  1  2  3
¯6 ¯4 ¯2  0  2  4  6
¯9 ¯6 ¯3  0  3  6  9

      +/X
28
      ×/X
5040
```

```
      (X-3)×(X-5)
8 3 0 ¯1 0 3 8
      R←8 7 6 5 4 3 2 1 0 ¯1
      M←R∘.=(X-3)×(X-5)

      M                          ' *'[1+M]
1 0 0 0 0 0 1                    *     *
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 1 0 0 0 1 0                     *   *
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 1 0 1 0 0                      * *
0 0 0 1 0 0 0                       *

      C
    ⊛      ⊛⊛⊛⊛    ⊛
   ⊛ ⊛     ⊛   ⊛   ⊛
  ⊛⊛⊛⊛⊛    ⊛⊛⊛⊛    ⊛
 ⊛     ⊛   ⊛       ⊛
⊛       ⊛  ⊛       ⊛⊛⊛⊛⊛

      (1 2 3 4 5-5)⌽C
        ⊛      ⊛⊛⊛⊛    ⊛
      ⊛ ⊛     ⊛   ⊛   ⊛
    ⊛⊛⊛⊛⊛    ⊛⊛⊛⊛    ⊛
  ⊛     ⊛   ⊛       ⊛
⊛       ⊛  ⊛       ⊛⊛⊛⊛⊛
```

| | NAME | SYMBOL | DEFINITION OR EXAMPLE | SECTION # |
|---|---|---|---|---|
| D | Addition | + | 3+4←→7 | 1.2 |
| Y | Multiplication | × | 3×4←→12 | 1.2 |
| A | Subtraction | - | 3-4←→¯1 | 3.1 |
| D | Division | ÷ | 3÷4←→.75 | 5.1 |
| I | Maximum | ⌈ | 3⌈4←→4 | 2.4 |
| C | Minimum | ⌊ | 3⌊4←→3 | 2.4 |
| | Power | ⋆ | 3⋆4←→81   A⋆B←→×/BρA | 2.5 6.5-6 |
| F | Remainder | \| | 3\|4←→1 | 7.1 |
| U | Relations | <≤=≥>≠ | 3<4←→1   4<3←→0 | 4.8 |
| N | Or | ∨ | | 14.2 |
| C | And | ∧ | ∨\|0 1   ∧\|0 1   ⍱\|0 1   ⍲\|0 1 | 14.2 |
| T | Not-or | ⍱ | 0\|0 1   0\|0 0   0\|1 0   0\|1 1 | 14.2 |
| I | Not-and | ⍲ | 1\|1 1   1\|0 1   1\|0 0   1\|1 0 | 14.2 |
| O | Domino | ⌹ | B⌹M is soln of B=M+.×X | 19.15 |
| N | Index-of | ⍳ | 2 3 5 7⍳5 2←→3 1 | 16.1 |
| S | Repetition | ρ | 3ρ5←→5 5 5 | 1.7 13.3 |
| | Catenation | , | 4 2,1 3 5←→4 2 1 3 5 | 6.2 |
| | Take | ↑ | 2↑4 5 6←→4 5 | 10.5 |
| | Drop | ↓ | 2↓4 5 6←→6 | 10.5 |
| | Compression | / | 0 1 1 0/1 2 3 4←→2 3 | 7.5 |
| M | Negation | - | -4←→¯4 | 8.2 |
| O | Reciprocal | ÷ | ÷4←→.25 | 8.3 |
| N | Magnitude | \| | \|¯4←→4 | 8.4 |
| A | Factorial | ! | !4←→1×2×3×4 | 8.1 |
| D | Ceiling | ⌈ | ⌈3.4←→4 | 8.5 |
| I | Floor | ⌊ | ⌊3.4←→3 | 8.5 |
| C | Complement | ~ | ~1←→0   ~0←→1 | 8.6 |
| | Matrix Inverse | ⌹ | M+.×⌹M is the identity | 19.15 |
| | Ravel | , | | 8.7 |
| | Integers | ⍳ | ⍳4←→1 2 3 4 | 1.5 |
| | Size | ρ | ρ4 1 3 6 2←→5 | 8.8 |
| | Flipping | ⌽ ⊖ ⍉ | Flip table about axis | 4.3 |
| O | Assignment | ← | X←6 | 1.3 |
| T | Indexing | X[I] | 2 3 5 7[2 4]←→3 7 | 4.4 |
| H | | M[I;J] | | |
| E | Function | ∇Z←F X | | 9.1-2 |
| R | Definition | ∇Z←X F Y | | |
| | Parentheses | | | 1.2 |
| | Execution order | | 3×4+5-7←→3×(4+(5-7)) | 1.2 |
| | Vectors | | 2 3 5×1 2 3←→2 6 15 | 1.6 |
| | Tables, Matrices | | | 2.1 13.3 |
| | Reduction (Over) | f/ | +/2 3 5←→10   ×/3 4←→12 | 1.4 4.10 |
| | Outer Product | ∘.f | | 2.3 |
| | Inner Product | f.g | | 13.2 13.4 |

SUMMARY OF NOTATION

# ALGEBRA:
# An
# Algorithmic Treatment

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# Preface

The present text treats the usual topics expected in a second course in algebra. It differs from conventional treatments in the following respects:

1. The notation used is simple and precise and applies to arrays (vectors and matrices) in a simple and uniform manner.

2. Arrays are used extensively to give a graphic view of functions by displaying the patterns produced by applying them to vectors. They are also used to clarify topics which use vectors directly, such as linear functions and polynomials.

3. The precision of the notation permits an algorithmic treatment of the material. In particular, every expression in the book can be executed directly by simply typing it on an appropriate computer terminal. Hence, if a computer is available it can be used by students for individual or collective exploration of relevant mathematical functions in the manner discussed in Appendix C. Even if a computer is not available, the algorithmic treatment presents the essentials of computer programming in a mathematical light, i.e., as the precise definition and application of functions.

4. The algorithmic approach is the same as that used in my <u>Elementary Functions</u> [3], a text which can be used as a continuation in topics such as the slope (derivative) of functions, and the circular, hyperbolic, exponential, and logarithmic functions.

5. The organization of topics follows a pattern suggested by considering algebra as a language; in particular, the treatment of formal identities is deferred until much work has been done in the reading and writing of algebraic sentences. These matters are discussed fully in Appendix A, "Algebra as a Language", and any teacher may be well-advised to begin by reading this appendix.

The pace of the text is perhaps best suited to a second course, but it can also be used in part for a first year course since the early chapters contain all of the essentials such as the introduction of the negative and rational numbers. When used for a second course these early chapters can serve not only as a brief review, but also as an introduction to the notation used.

The text employs the APL language which is available on computer terminals. Although an APL computer is in no way essential, it can be a very useful adjunct. Moreover, the text can be used to provide interesting material and exercises for courses devoted to introducing APL itself. Finally, the text should be useful in a variety of algebra courses in both high school and college, since it presents traditional material in a new light, combined with the interest of learning to program and use a computer.

This text grew out of a summer project undertaken in 1969 in collaboration with my colleagues Adin Falkoff and Paul Berry of IBM, and with five high school teachers - Mr. John Brown, now of Dawson College, Montreal; Mr. Nathaniel Bates, of Belmont Hill School, Belmont, Massachusetts; Miss Linda Alvord, of Scotch Plains-Fanwood High School, Scotch Plains, New Jersey; and Sisters Helen Wilxman and Barbara Brennan, of Mary Immaculate School, Ossining, New York. Mr. Peter Manchester provided valuable assistance in preparing APL programs and in developing exercises. I am indebted to all of these people for much fruitful discussion, and particularly to Messrs. Falkoff and Berry for helping to set and maintain the direction of the project.

I have also benefitted greatly from discussions with Miss Nancy Boyd and Mr. Christopher Edley, students at Swarthmore College. These discussions arose from their work as tutors in summer courses using this text which were presented at Swarthmore College by Professor David Rosen and by Mr. Russell Daniel, now of Temple University. I must also acknowledge many helpful discussions with colleagues of the Philadelphia Scientific Center of IBM, particularly Messrs. E. E. McDonnell and P. C. Berry, as well as critical detailed reviews of the text by Miss Alvord and Mrs. Sandra Pakin. I am also grateful for the support of the IBM Corporation, particularly for the freedom provided by its Fellow program.

The manuscript was entered, revised, and printed on an APL text editor system. For outstanding clerical assistance in the use of this system I am indebted to my wife Jean, and to Mrs. Susan O'Connell. The artwork was done by Mr. David Hatcher, who also worked long hours in the production of the final draft. I am particularly indebted to Miss Elizabeth Llanso for her patient and unfailing assistance in every aspect of the preparation of the manuscript.

*July, 1972* Kenneth E. Iverson
*Philadelphia, Pennsylvania*

# Contents

# Illustrations

# 1

# The Language of Mathematics

## 1.1 INTRODUCTION

Algebra is the language of mathematics. It is therefore an essential topic for anyone who wishes to continue the study of mathematics. Moreover, enough of the language of algebra has crept into the English language to make a knowledge of some algebra useful to most non-mathematicians as well. This is particularly true for people who do advanced work in any trade or discipline, such as insurance, engineering, accounting, or electrical wiring. For example, instructions for laying out a playing field might include the sentence, "To verify that the corners are square, note that the length of the diagonal must be equal to the square root of the sum of the squares of the length and the width of the field", or alternatively, "The length of the diagonal must be $\sqrt{\ell^2+w^2}$". In either case (whether expressed in algebraic symbols or in the corresponding English words), the comprehension of such a sentence depends on a knowledge of some algebra.

Because algebra is a language, it has many analogies with English. These analogies can be helpful in learning algebra, and they will be noted and explained as they occur. For instance, the integers or counting numbers (1, 2, 3, 4, 5, 6...) in algebra correspond to the concrete nouns in English, since they are the basic things we discuss, and perform operations upon. Furthermore, functions in algebra (such as + (plus), - (subtract), and × (times)) correspond to the verbs in English, since they do something to the nouns. Thus, $2+3$ means "add 2 to 3", and $(2+3)\times 4$ means "add 2 to 3 and then multiply by 4". In fact, the word "function" (as defined, for example, in the American Heritage Dictionary), is descended from an older word meaning, "to execute", or "to perform".

When the language of algebra is compared to the language of English, it is in certain respects much simpler, and in other respects more difficult. Algebra is simpler in that the basic algebraic sentence is an instruction to do something, and algebraic sentences (usually called expressions) therefore correspond to imperative English sentences (such as "Close the door."). For example, $2+3$

means "add 2 and 3", and $YEAR \leftarrow 1970$ means "assign to the name $YEAR$ the value 1970", and $Y \leftarrow 1970$ means "assign to the name $Y$ the value 1970". Since imperative sentences form only a small and relatively simple part of English, the language of algebra is in this respect much simpler.

Algebra is also simpler in that it permits less freedom in the ways you can express a particular function. For example, "subtract 2 from 4" would normally be written in algebra only as $4-2$, whereas in English it could be expressed in many ways such as "take the number 2 and subtract it from the number 4", or "compute the difference of the integers 4 and 2".

The most difficult aspect of traditional presentations of algebra is the early emphasis on <u>identities</u>, or the equivalence of different expressions. For example, the expressions $(5+7)\times(5+7)$ and $(5\times5)+(2\times5\times7)+(7\times7)$ are <u>equivalent</u> in the sense that, although they involve a different sequence of funtions, they each yield the same result. English also offers equivalent expressions. For example, "The dog bit the man" is equivalent to "The man was bitten by the dog". It is not that the rules for determining equivalence in algebra are more difficult than in English; on the contrary, they are so much simpler that their study is more rewarding and therefore more attention is given to equivalences in algebra than in English.

In the present treatment this aspect of algebra (that is, the study of identities or equivalence of expressions), is delayed until the student has devoted more attention to the reading, writing, and evaluation of algebraic expressions.

This view of algebra as a language is central to the present treatment. It is buttressed and expanded in Appendix A, and this appendix should perhaps be read first by any teacher and by any student who has significant prior experience with traditional treatments of algebra.

The exercises form an important part of the development, and the point at which the reader should be prepared to attempt each group of exercises is indicated in the margin. For example, the first such marginal note appears as ⌹1-6 and indicates that Exercises 1 to 6 of this chapter may be attempted at that point.

Collections of expressions occurring in certain exercises are broken into groups to provide convenient reference in assigning and discussing exercises. These groups sometimes indicate substantive groupings of the material treated as well.

The exposition and the exercises are organized to encourage experimentation and observation as an essential part of learning. Experimentation and discovery can be further encouraged to a startling degree by the use of an APL computer terminal if one is available. All expressions occurring in the text and exercises can be entered directly on the terminal keyboard without further knowledge of computers. Techniques for the use of the computer in teaching are discussed in Appendix C. Appendix B presents the computer keyboard and other details necessary to putting it in operation.

A student using an APL computer in exploration is sometimes confronted with matters not treated until a later point in the text. For example, a beginning student entering the expression

```
      2000×3000×4000
```

will receive the response

```
2.4E10
```

This result is expressed in exponential notation (meaning 2.4 times 10 to the power 10) which is not discussed until Section 5.17. the Index, the Summary of Notation (appearing inside the covers), and Appendix B can be used to resolve such difficulties.

## 1.2 EXPRESSIONS AND RESULTS

Evaluation of the expression 2+3 produces the result 5. Such a fact will be written in the following form:

```
      2+3
5
```

and will be read aloud as "2 plus 3 makes 5". The following examples would be read in a similar way:

```
      7+12
19

      8×4
32
```

Where there is more than one function to be executed, parentheses are used to indicate which is to be done first. Thus the expression

```
      (2+3)×4
```

is evaluated by first performing the function within the parentheses (that is, 2+3), and then multiplying the result by 4. The final result is therefore 20, as shown below:

```
      (2+3)×4
20
```

The foregoing is read aloud as "quantity 2+3, times 4". The word "quantity" indicates that the first expression following it is to be executed first. That is, you are to find the result of 2+3 before attempting to execute the function "times".

The steps in the execution of an expression may be displayed on successive lines, substituting at each line the value of part of the expression above it as illustrated below:

```
     |(2+3)×4
     |
     |  5  ×4
20
```

The vertical line drawn to the left of the first two lines indicates that they are <u>equivalent</u> statements, either of which would produce the result 20 shown on the final line. The whole statement would be read aloud as "Quantity 2 plus 3 times 4 is equivalent to 5 times 4 which makes 20". The following examples would be read in a similar way as shown on the right:

```
     |(2+3)×(5+4)          Quantity 2 plus 3 times quantity 5
     |                        plus 4
     |                     is equivalent to
     |  5  ×  9            5 times 9
45                         which makes 45

     |((2×3)+(5×4))×2      Quantity 2 times 3 plus quantity 5
     |                        times 4, all times 2
     |                     is equivalent to
     |(   6  +  20  )×2    quantity 6 plus 20 times 2
     |                     is equivalent to
     |      26       ×2    26 times 2
52                         which makes 52
```

The last example illustrates the difficulty of expressing in English the sequence of execution that is expressed so simply by parentheses in algebra, that is, when parentheses are "nested" within other parentheses even the use of the word "quantity" does not suffice and one resorts to expressions such as "all times 2". The main point is this: in learning any new language (such as algebra) it is important to re-express statements in a more familiar language (such as English); however, certain things are so awkward to express in the old language that it becomes important to learn to "think" in the new language. ⌹1-6

The expression 2+3×4, written without parentheses, could be taken to mean either (2+3)×4 (which makes 20), or 2+(3×4) (which makes 14). To avoid such ambiguity we make the following rule: when two or more functions occur in succession with no parentheses between them, the rightmost function is executed first. For example:

```
      |2+3×4
      |
      |2+ 12
14
      |1+2×3+4×5
      |
      |1+2×3+ 20
      |
      |1+2× 23
      |
      |1+  46
47
      |(1+2×3)+4×5
      |
      |(1+ 6)+ 20
      |
      |   7  +20
27
```
⌹7-12

## 1.3 NAMES

Consider the following statements:

```
      (1+3+5+7+9)×2
50
      (1+3+5+7+9)×3
75
      (1+3+5+7+9)×4
100
```

Since the expression 1+3+5+7+9 occurs again and again in the foregoing statements, it would be convenient to give some short name to the result produced by the expression, and then use that short name instead of the expression. This is done as follows:

```
      IT←1+3+5+7+9
      IT×2
50
      IT×3
75
      IT×4
100
      IT
25
```

The foregoing would be read aloud as follows: "The name *IT* is assigned the value of the expression 1+3+5+7+9. *IT* times 2 makes 50. *IT* times 3 makes 75. *IT* times 4 makes 100. *IT* makes 25".

Names can be chosen at will. For example:

```
      LENGTH←5
      WIDTH←4
      LENGTH×WIDTH
20
      AREA←LENGTH×WIDTH
      AREA
20
      PRICE←5
      QUANTITY←4
      PRICE×QUANTITY
20
```

Mathematicians usually prefer to use short names like *L* or *W* or *X* or *Y*, perhaps because this brings out the underlying structure or similarity of expressions which may deal with different names. Consider, for example, the following sequence:

```
      X←5
      Y←4
      X×Y
20
```

If *X* is taken to mean length and *Y* is taken to mean width, the result is the area of the corresponding rectangle; but if *X* is taken to mean price and *Y* is taken to mean quantity, then the result is the total price. This makes clear that there is some similarity between the calculation of an area from length and width and the calculation of total price from price and quantity.

The names used in algebra are also called variables, since they may vary in the sense that the same name may represent different values at different times. For example:

```
      X←3
      X×X
9
      X←5
      X×X
25
```

This ability to vary distinguishes a name like *X* from a symbol like 5 which always represents the same value and is therefore called a constant.

It is interesting to note that the variables in algebra correspond to the pronouns in English. For example, the sentence "close it" is meaningless until one knows to what "it" refers. This reference is usually made clear by a preceding sentence. For example, "See the door. Close it" is unambiguous because the first sentence makes clear that "it" refers to "the door". Similarly, in algebra the expression *IT*+5 cannot be evaluated unless the value to which *IT* refers is known. In algebra this reference is made clear in one way, by the use of the assignment represented by the symbol ←. For example:

```
      IT←3
      IT+5
8
```

The same name *IT* can stand for different values at different times just as the pronoun "it" can refer to different things at different times. ⌹13-18

## 1.4 OVER NOTATION

It is often necessary to take the sum over a whole list of numbers. For example, if the list consists of the numbers 1 3 5 7 9 11, then their sum could be written as

```
      1+3+5+7+9+11
36
```

It is more convenient to use the following notation:

```
      +/1 3 5 7 9 11
36
```

The foregoing is read aloud as "Sum over 1 3 5 7 9 11" or as "Plus over 1 3 5 7 9 11".

The <u>over</u> notation can be used for other functions as well as for addition. For example:

```
                                  READ AS

      ×/1 2 3                     Times over 1 2 3
6                                 makes 6

      ×/1 2 3 4                   Times over 1 2 3 4
24                                makes 24

      +/1 2 3 4                   Plus over 1 2 3 4
10                                makes 10

      (+/1 2 3 4)×6               Quantity plus over 1 2 3 4
                                    times 6
60                                makes 60

      6× +/1 2 3 4                6 times plus over 1 2 3 4
60                                makes 60

      N←1 2 3 4                   N assigned 1 2 3 4
      +/N                         Plus over N
10                                makes 10

      ×/N                         Times over N
19-21 24                          makes 24
```

## 1.5 THE POSITIVE INTEGERS

The natural numbers 1 2 3 4 5 . . . are also called the <u>positive integers</u>. They may be produced as follows:

```
      ⍳3
1 2 3
      ⍳5
1 2 3 4 5
      ⍳16
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
      N←6
      ⍳N
1 2 3 4 5 6
```

The symbol ⍳ is the Greek letter <u>iota</u> which corresponds to the English letter i. The expression ⍳*N* is read aloud as "the integers to *N*". Thus:

```
                                  READ AS

      +/⍳5                        Plus over the integers to 5
15                                makes 15

      ×/⍳5                        Times over the integers to 5
22-25 120                         makes 120
```

## 1.6 VECTORS

A list of numbers such as `3 5 7 11` is called a <u>vector</u>. The numbers in the list are called the <u>elements</u> of the vector. Thus the first element of the vector `3 5 7 11` is the number `3`, the second element is `5`, the third element is `7`, and the fourth is `11`. The number of elements in the vector is called the <u>size</u> of the vector. Thus the size of the vector `3 5 7 11` is `4`. Any single quantity (such as `17`) will now be referred to as a <u>scalar</u> to distinguish it from a vector.

Vectors can be added and multiplied as shown in the following examples:

```
                    READ AS

      3 5 7+1 2 3   Vector 3 5 7 plus vector 1 2 3
4 7 10              makes 4 7 10

      1 2 3+3 2 1   Vector 1 2 3 plus vector 3 2 1
4 4 4               makes 4 4 4

      1 2 3×3 2 1   Vector 1 2 3 times 3 2 1
3 4 3               makes 3 4 3
```

From this it should be clear that when two vectors are added the first element is added to the first element, the second element is added to the second, and so on. Multiplication is performed similarly.

Like any other result, a vector can be assigned a name. For example:

```
                    READ AS

      V←1 2 3 4     The name V is assigned vector 1 2 3 4

      W←4 3 2 1     The name W is assigned vector 4 3 2 1

      V+W           V plus W
5 5 5 5             makes 5 5 5 5

      V×W           V times W
4 6 6 4             makes 4 6 6 4

      V×V           V times V
1 4 9 16            makes 1 4 9 16
```

The following examples may be read similarly:

```
                      READ AS

      N←⍳5            N is assigned integers to 5

      N               N
1 2 3 4 5             makes 1 2 3 4 5

      N×N             N times N
1 4 9 16 25           makes 1 4 9 16 25

      (⍳6)×⍳6         Quantity integers to 6 times
                        quantity integers to 6
1 4 9 16 25 36        makes 1 4 9 16 25 36
```

Since the addition of two vectors *V* and *W* means that the first element of *V* is to be added to the first element of *W*, the second element of *V* is to be added to the second element of *W*, and so on, then an expression such as

```
      1 3 5+6 8 1 4 3
```

cannot be executed because the vectors are not of the same size. However, expressions of the following form <u>can</u> be executed:

```
                      READ AS

      3+1 3 5 7       3 plus vector 1 3 5 7
4 6 8 10              makes 4 6 8 10

      1 2 3 4 5+6     Vector 1 2 3 4 5 plus 6
7 8 9 10 11           makes 7 8 9 10 11
```

In other words, if one of the quantities to be added is a single number (i.e., a scalar), it is added to each element of the vector. The same holds for multiplication as follows:

```
                      READ AS

      3×1 3 5 7       3 times vector 1 3 5 7
3 9 15 21             makes 3 9 15 21

      3×⍳5            3 times integers to 5
3 6 9 12 15           makes 3 6 9 12 15

      2+3×⍳5          2 plus 3 times integers to 5
5 8 11 14 17          makes 5 8 11 14 17

      1+2×⍳6          1 plus 2 times integers to 6
3 5 7 9 11 13         makes 3 5 7 9 11 13
```

```
      +/1+2×⍳6        plus over 1 plus 2 times integers to 6
48                    makes 48

      1++/1+2×⍳6      1 plus plus over 1 plus 2 times
                        integers to 6
49                    makes 49
```

⌹26-28

## 1.7 REPETITIONS

Consider the following statements and their verbalization:

```
                      READ AS

      3⍴2             3 repetitions of 2
2 2 2                 makes 2 2 2

      2⍴3             2 repetitions of 3
3 3                   makes 3 3

      5⍴7             5 repetitions of 7
7 7 7 7 7             makes 7 7 7 7 7
```

The symbol ⍴ is the Greek letter <u>rho</u> which corresponds to the English r.

The following two columns of statements show some interesting properties of repetitions, including the relation between multiplication and a sum of repetitions:

```
      +/3⍴2                 2×3
6                     6

      +/4⍴2                 2×4
8                     8

      +/5⍴7                 7×5
35                    35

      +/15⍴20               20×15
300                   300

      ×/2⍴2                 ×/2⍴3
4                     9

      ×/3⍴2                 ×/3⍴3
8                     27

      ×/4⍴2                 ×/4⍴3
16                    81

      ×/5⍴2                 ×/5⍴3
32                    243
```

⌹29-31

## 1.8 SUMMARY

This chapter has been concerned primarily with the language or notation of algebra, and the uses of the notation have been kept simple. Now that the language has been mastered, succeeding chapters can turn to more interesting uses of it. This does not imply that <u>all</u> the notation of algebra has now been covered, but rather that the main ideas have been introduced and that any further additions will be easy to grasp. The situation may be compared to the learning of a natural language such as French. Once the main ideas of the language have been learned (in months or years of study), the new French words needed for some particular purpose can be picked up more easily.

For example, the next chapter treats the <u>maximum</u> function, represented by the symbol ⌈ and defined to yield the larger of its two arguments:

```
                      READ AS

      2⌈3             2 maximum 3
3                     makes 3

      2⌈4             2 maximum 4
4                     makes 4

      2⌈5             2 maximum 5
5                     makes 5

      5⌈2             5 maximum 2
5                     makes 5
```

The important point is that this new function is treated exactly like the functions plus and times, thus:

```
      2⌈1 2 3 4
2 2 3 4

      3⌈⍳5
3 3 3 4 5

      ⌈/8 1 7 10 3 10
10

      1 2 3 4 5⌈5 4 3 2 1
5 4 3 4 5
```

The main points of the notation introduced in this chapter will now be summarized in a few examples which should be useful for reference purposes:

| EXAMPLE | READ AS | COMMENTS |
|---|---|---|
| (2+3)×4<br>20 | Quantity 2 plus 3 times 4<br>makes 20 | Function in parentheses is executed first |
| 2+3×4<br>14 | 2 plus quantity 3 times 4<br>makes 14 | Rightmost function is executed first if there are no intervening parentheses |
| *N*←3 | *N* is assigned 3 | Name *N* is assigned the value of the expression to the right of ← |
| *N*×4<br>12 | *N* times 4<br>makes 12 | |
| +/3 5 7<br>15 | Plus over vector 3 5 7<br>makes 15 | |
| ×/2 3 5 2<br>60 | Times over vector 2 3 5 2<br>makes 60 | |
| 1 2 3×3 2 1<br>3 4 3 | Vector 1 2 3 times vector 3 2 1<br>makes 3 4 3 | Element-by-element multiplication |
| 3×1 2 3<br>3 6 9 | 3 times vector 1 2 3<br>makes 3 6 9 | Single number multiplies each element |
| ⍳5<br>1 2 3 4 5 | Integers to 5<br>makes 1 2 3 4 5 | |
| 5⍴4<br>4 4 4 4 4 | 5 repetitions of 4<br>makes 4 4 4 4 4 | ⌹32 |

# 2

# Function Tables and Maps

## 2.1 INTRODUCTION

In Chapter 1, addition was spoken of as a "function" because it "does something" to the numbers it is applied to and produces some result. Multiplication was also referred to as a function, but the notion of function is actually much broader than these two examples alone might suggest. For example, the average or *normal* weight of a woman depends on her height and is therefore a function of her height. In fact, if one were told that the normal weight for a height of 57 inches is 113 pounds, the normal weight for a height of 58 inches is 115 pounds, and so on, then one could evaluate the function "normal weight" for any given height by simply consulting the list of corresponding heights and weights.

It is usually most convenient to present the necessary information about a function such as "normal weight" not by a long English sentence as begun above, but by a *table* of the form shown in Figure 2.1.

| | | | |
|---|---|---|---|
| H | 57 | 113 | W |
| E | 58 | 115 | E |
| I | 59 | 117 | I |
| G | 60 | 120 | G |
| H | 61 | 123 | H |
| T | 62 | 126 | T |
| | 63 | 130 | |
| I | 64 | 134 | I |
| N | 65 | 137 | N |
| | 66 | 141 | |
| I | 67 | 145 | P |
| N | 68 | 149 | O |
| C | 69 | 153 | U |
| H | 70 | 157 | N |
| E | 71 | 161 | D |
| S | 72 | 165 | S |

Table of Normal Weights
Versus Heights

Figure 2.1

The quantity (or quantities) to which a function is applied is (are) called the *argument* (or *arguments*) of the function. For example, in the expression 3×4 the number 3 is the *left* (or *first*) argument of the function × and 4 is the *right* (or *second*) argument. Evaluation of the "normal weight" function (represented by Table 2.1) for a given argument (say 68 inches) is performed by finding the argument 68 in the first column and reading the weight (149 pounds) which occurs in the same row.

The *domain* of a function is the collection of all arguments for which it is defined. Addition is, of course, defined for any pair of numbers, but the function "normal weight" is certainly not defined for heights such as 2 inches or 200 inches. For practical purposes, the domain of a function such as "normal weight" is simply the collection of arguments in the table we happen to possess, even though information for other arguments might be available elsewhere. For example, the domain of the function of Table 2.1 is the set of integers from 57 to 72, that is, the set of integers 56 +ι16.

The *range* of a function is the collection of all the results of the function. For example, the range of the function of Figure 2.1 is the set of integers 113, 115, 117 120, etc., occurring in the second column. 1-2

A table of normal weights often shows several columns of weights, one for small framed people, one for mediumn, and one for large. Such a table appears in Figure 2.2. In such a case the weight is a function of two arguments, the height and the "frame-class"; the first argument determines the row and the second argument determines the column in which the result appears. Thus the normal weight of a small-boned, 66-inch woman is 133 pounds. 3-4

An arithmetic function can also be represented by a table, as is illustrated by Figure 2.3 for the case of multiplication. Since the domain of multiplication includes all numbers, no table can represent the entire multiplication function; Figure 2.3, for example, applies only to the domain of the first few integers. The multiplication sign in the upper left corner is included simply to indicate the arithmetic function which the table represents.

In any table, the first column represents the domain of the first argument and the first row represents the domain of the second argument; the rest is called the *body* of the table. For example, in Figure 2.3, the body of the table is that part bordered on the left and top by the solid lines.

| | | Frame | | | |
|---|---|---|---|---|---|
| | | Small | Medium | Large | |
| H | 57 | 105 | 113 | 121 | W |
| E | 58 | 107 | 115 | 123 | E |
| I | 59 | 109 | 117 | 125 | I |
| G | 60 | 112 | 120 | 128 | G |
| H | 61 | 115 | 123 | 131 | H |
| T | 62 | 118 | 126 | 135 | T |
| | 63 | 122 | 130 | 139 | |
| I | 64 | 126 | 134 | 143 | I |
| N | 65 | 129 | 137 | 147 | N |
| | 66 | 133 | 141 | 151 | |
| I | 67 | 137 | 145 | 155 | P |
| N | 68 | 141 | 149 | 158 | O |
| C | 69 | 145 | 153 | 162 | U |
| H | 70 | 149 | 157 | 165 | N |
| E | 71 | 153 | 161 | 169 | D |
| S | 72 | 157 | 165 | 173 | S |

Normal Weight as a Function
of Two Arguments

Figure 2.2

| | | Right Domain | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Function Name | × | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| | 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| Left | 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| Domain | 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| | 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| | 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| | 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |

Multiplication Table

Figure 2.3

In any table representing a function of two arguments, any one column of the body (taken together with the column of arguments not in the body) represents a function of one argument. For example, if one takes the second column of the <u>body</u> of Figure 2.2, it represents the same function of one argument as does Figure 2.1.

Thus any function of two arguments can be thought of as a collection of functions of one argument. For example, the second column of the body of Figure 2.3 represents the "times two" function, the third column represents the "times three" function, etc.

Similarly, one row of the body of a function table represents a function of one argument. For example, the fifth row of the body of Figure 2.2 gives weights as a function of "frame" for 61 inch women. 5-10

## 2.2 READING FUNCTION TABLES

The basic rule for reading a function table is very simple: to evaluate a function, find the row in which the value of the first argument occurs (in the first column, not in the body of the table) and find the column in which the second argument occurs (in the first row) and select the element at the intersection of the selected row and the selected column. However, just as there is more to reading an English sentence than pronouncing the individual words, so a table can be "read" so as to yield useful information about a function beyond that obtained by simply evaluating it for a few cases.

For example, the table of Figure 2.2 can be "read" so as to answer the following questions:

1. Can two women of different heights have the same normal weight?

2. For a given frame type, does normal weight always increase with increasing height?

3. For a given height, does normal weight increase with frame type?

4. How many inches of height produce (about) the same change in weight as the change from small to large frame? Does this change remain about the same throughout the table?

Arithmetic functions are more orderly than a function such as that represented by Figure 2.2, and the patterns that can be detected in reading their function tables are more striking and interesting. Consider, for example, an attempt to read Figure 2.3 to answer the following questions:

5. The second column of the body (which was previously remarked to represent the "times two" function) contains the numbers 2 4 6, etc., which are encountered in "counting by twos". Can a similar statement be made about the other columns?

6. Is there any relation between corresponding rows and columns of the body, e.g., between the third row and the third column?

7. Can every result in the body be obtained in at least two different ways? Are there any results which can be obtained in only two ways?

Similarly, one can construct a function table for addition and read it to determine answers to the following questions:

8. In how many different ways can the result 6 be obtained by addition? Does the result 6 occur in the table in some pattern and if so does a similar pattern apply to other results such as 7, 8, etc.?

9. What is the relation between two successive rows of the table?

Because of the patterns they exhibit, function tables can be very helpful in gaining an understanding of unfamiliar mathematical functions. For this reason they
11⊞ will be used extensively in succeeding chapters.

## 2.3 EXPRESSIONS FOR PRODUCING FUNCTION TABLES

If

```
      A←1 2 3 4 5 6 7 8
      B←1 2 3 4 5 6 7 8 9 10
```

then the expression `A∘.×B` yields the body of the function table of Figure 2.3 as follows:

```
      A∘.×B
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |

Similarly, the body of an addition table for the same set of arguments can be produced as follows:

```
      A∘.+B
```

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

The general rule is that the symbol ∘ (called <u>null</u>) followed by a period followed by the symbol for a function produces the appropriate function table when applied to any arguments *A* and *B*. The expression "*A*∘.+*B*" may be read as "the addition table for *A* and *B*" or as "*A* addition table *B*", or even as "*A* null dot plus *B*". Similarly, "*A*∘.×*B*", may be read as "*A* times table *B*", etc.

It is important to note that the expression *A*∘.+*B* produces only the <u>body</u> of the addition table to which one may add a first column consisting of *A* and a first row consisting of *B* if this is found to make the table easier to read.

It is also important to note the difference between the expression *A*∘.×*B*, which yields the multiplication table, and the expression *A*×*B*, which yields the element-by-element product of *A* and *B*. For example:

```
      A←1 3 5
      B←2 4 6

      A×B
2  12  30

      A∘.×B
   2   4   6
   6  12  18
  10  20  30
```

⌹12-13

The body of a table alone does not define a function. For example, the following tables define two distinct functions although the bodies of the tables are identical:

| + | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 2 | 4 | 5 | 6 | 7 |
| 3 | 5 | 6 | 7 | 8 |
| 4 | 6 | 7 | 8 | 9 |
| 5 | 7 | 8 | 9 | 10 |

| *F* | 2 | 3 | 5 | 7 |
|---|---|---|---|---|
| 6 | 4 | 5 | 6 | 7 |
| 5 | 5 | 6 | 7 | 8 |
| 4 | 6 | 7 | 8 | 9 |
| 3 | 7 | 8 | 9 | 10 |

The name of the function represented by the first table is + (as shown in the upper left corner), and the table can be used to evaluate expressions as shown on the left below:

```
5 + 3 is 8          5 F 3 is 6
4 + 5 is 9          4 F 5 is 8
3 + 3 is 6          3 F 3 is 8
```

The function represented by the second table is called *F* (as indicated in the upper left corner) and the expressions on the right above show the evaluation of the function *F* for the same arguments used on the left. Since the results differ, the two tables represent different functions.

The complete specification of a function table therefore requires the specification of four items:

1. The left domain (i.e., the domain of the left argument).

2. The right domain.

3. The body of the table.

4. The name of the function.

From these four items the table can be constructed and used as illustrated below:

```
Left domain:    3 4 5 6
Right domain:   11 9 7 5 3 1
Body:           5 + (3×⍳4)∘.+(2×⍳6)
Name:           G
```

| *G* | 11 | 9 | 7 | 5 | 3 | 1 |
|---|---|---|---|---|---|---|
| 3 | 10 | 12 | 14 | 16 | 18 | 20 |
| 4 | 13 | 15 | 17 | 19 | 21 | 23 |
| 5 | 16 | 18 | 20 | 22 | 24 | 26 |
| 6 | 19 | 21 | 23 | 25 | 27 | 29 |

4 *G* 5 is 19
6 *G* 9 is 21
14-16⊞ 2×6 *G* 9 is 42

## 2.4 THE FUNCTIONS DENOTED BY ⌈ AND ⌊

The advantages of the function table can perhaps be better appreciated by applying it to some unfamiliar functions than by applying it to functions such as addition and multiplication which are probably already well understood by the reader. For this purpose we will now introduce several simple new functions which will also be found to be very useful in later work.

It is sometimes instructive to introduce a new function as a puzzle - the reader must determine the general rule for evaluating the function by examining the results obtained when it is applied to certain chosen arguments. For example, the function ⌊ can be applied to certain arguments with the results shown below:

```
      3⌊8
3
      47⌊32
32
```

If one performs enough such experiments it should be possible to guess the general rule for the function. In attempting such a guess it is helpful to organize the experiments in some systematic way, and the body of the function table provides precisely the sort of organization needed. For example:

```
      I←1 2 3 4 5 6 7 8
      I∘.⌊I
1  1  1  1  1  1  1  1
1  2  2  2  2  2  2  2
1  2  3  3  3  3  3  3
1  2  3  4  4  4  4  4
1  2  3  4  5  5  5  5
1  2  3  4  5  6  6  6
1  2  3  4  5  6  7  7
1  2  3  4  5  6  7  8
```

From the foregoing the reader should be able to state the definition of the function and from that statement be able to apply it correctly to any pair of arguments.

The function ⌊ is called the minimum function because it yields the smaller of its two arguments. The maximum function is denoted by ⌈ and is defined analogously. The body of its function table appears below:

```
      I∘.⌈I
1  2  3  4  5  6  7  8
2  2  3  4  5  6  7  8
3  3  3  4  5  6  7  8
4  4  4  4  5  6  7  8
5  5  5  5  5  6  7  8
6  6  6  6  6  6  7  8
7  7  7  7  7  7  7  8
8  8  8  8  8  8  8  8
```

⌹17-18

## 2.5 THE POWER FUNCTION

Another very useful function is called the power function and is denoted by *. The body of its function table is shown below:

```
      I←1 2 3 4 5 6 7
      I∘.*I
 1   1    1     1      1       1       1
 2   4    8    16     32      64     128
 3   9   27    81    243     729    2187
 4  16   64   256   1024    4096   16384
 5  25  125   625   3125   15625   78125
 6  36  216  1296   7776   46656  279936
 7  49  343  2401  16807  117649  823543
```

The power function is defined in terms of multiplication in much the same way as multiplication is defined in terms of addition. To appreciate how multiplication is defined as "repeated additions", consider the following expressions:

```
      2⍴2
 2  2
      +/2⍴2                     2×2
 4                      4
      3⍴2
 2  2  2
      +/3⍴2                     2×3
 6                      6
      4⍴2
 2  2  2  2
      +/4⍴2                     2×4
 8                      8
      +/5⍴2                     2×5
10                     10
      +/6⍴2                     2×6
12                     12
      +/8⍴3                     3×8
24                     24
```

Comparing the results of +/2⍴2 and 2×2 and the results of +/3⍴2 and 2×3, etc., it should be clear that *M*×*N* is equivalent to adding *N* quantities each having the value *M*.

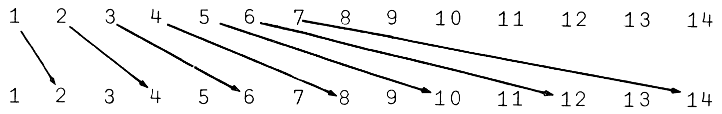
The corresponding definition of the power function ⋆ can be obtained by replacing each occurrence of + in the foregoing expressions by × and each occurrence of × by ⋆:

```
      2⍴2
2  2
      ×/2⍴2                     2⋆2
4                       4
      3⍴2
2  2  2
      ×/3⍴2                     2⋆3
8                       8
      4⍴2
2  2  2  2
      ×/4⍴2                     2⋆4
16                     16
      ×/5⍴2                     2⋆5
32                     32
      ×/6⍴2                     2⋆6
64                     64
      ×/8⍴3                     3⋆8
6561                 6561
```

In general, *M* to the power *N* (that is, *M*⋆*N*) is obtained by multiplying together *N* factors each having the
19-22 value *M*.

## 2.6 MAPS

Figure 2.4 shows a <u>map</u> which represents the "times two" function. The rule for evaluating a function represented by a map is very simple: locate the specified argument in the top row, then follow the arrow from that argument to the result at the head of the arrow in the bottom row; e.g., the result for the argument 3 is 6.

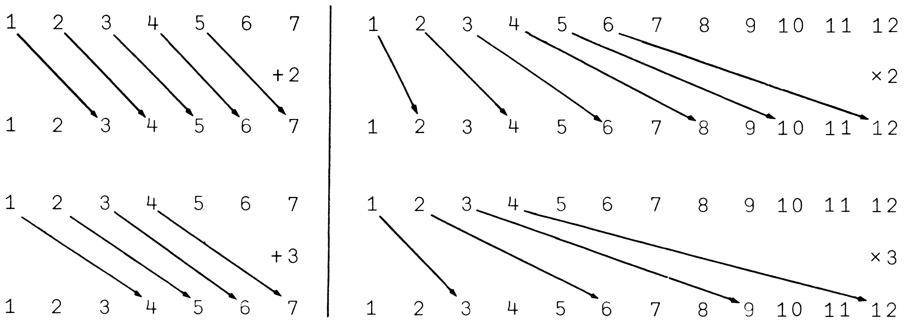



Map of "Times Two" Function

Figure 2.4

The rules for constructing a map are also simple. First consider all of the values in the domain of the function together with all of the results, and choose the smallest number and the largest number from among them. Write a row of numbers beginning with the smallest and continuing through each of the integers in order up to the largest. Repeat the same numbers in a row directly below the first row. For each argument in the top row now draw an arrow to the corresponding result in the bottom row.

Just as it is often helpful to read tables, so is it helpful to read such maps. Consider the four maps shown in Figure 2.5. From the first it is clear that in the map of addition of 2, the arrows are all parallel. From the map below this it is clear that the same is true for addition of 3, and that the slope of the arrows depends on the amount added. The maps on the right show multiplication; here the arrows are not parallel, and the distance between successive arrowheads is seen to be equal to the multiplier.



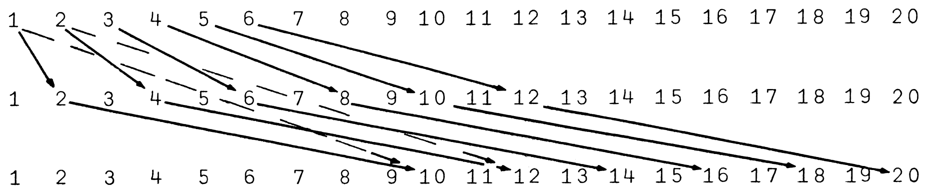

Maps for Addition and Multiplication

Figure 2.5

It is sometimes useful to show the maps of a sequence of functions such as the following:

```
      I←1 2 3 4 5 6
      2×I
2  4  6  8  10  12
      8+(2×I)
10  12  14  16  18  20
```

The appropriate maps are shown in Figure 2.6. The broken lines show the map of the overall result produced, that is, the map of the function 8+(2×*I*).




Maps of a Sequence of Functions

Figure 2.6

Maps will be used in the next chapter to introduce the function subtraction and the new negative numbers which this
23-24 function produces.

# 3

# The Negative Numbers

## 3.1 SUBTRACTION

The <u>subtraction</u> function is denoted by the <u>minus</u> sign (-). For example:

```
                            READ AS

      8-3                   8 minus 3
5                           makes 5
      (5+3)-3               Quantity 5+3 minus 3
5                           makes 5
      (5-3)+3               Quantity 5-3 plus 3
5                           makes 5
```

The following examples illustrate the relation between addition and subtraction:

```
      5+3                         8-3
8                           5
      6+3                         9-3
9                           6
      7+3                         10-3
10                          7
      5+4                         9-4
9                           5
      6+4                         10-4
10                          6
      7+4                         11-4
11                          7

      1 2 3 4 5+3                 4 5 6 7 8-3
4  5  6  7  8               1  2  3  4  5
      1 2 3 4 5+4                 5 6 7 8 9-4
5  6  7  8  9               1  2  3  4  5
```

From these examples it appears that subtraction will undo the work of addition. That is, if 3 is added to 5 to produce 8, and 3 is then subtracted from 8 the final result is the original value 5. This is true in general, and subtraction is therefore said to be the <u>inverse</u> of addition. Thus for any number $X$ and any number $A$, the expression $(X+A)-A$ will yield $X$.

The converse is also true; that is, addition will undo the work of subtraction, and addition is therefore the inverse of subtraction. For example:

```
      8-3
5
      5+3
8
      8 9 10 11 12 13-3
5 6 7 8 9 10
      5 6 7 8 9 10+3
8 9 10 11 12 13
```

In other words, $(X-A)+A$ will also yield $X$.

In summary then:

$(X+A)-A$ makes $X$

$(X-A)+A$ makes $X$

For example:

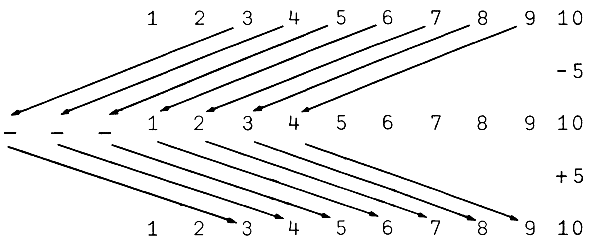
```
      (8 9 10 11 12 13+3)-3
8 9 10 11 12 13
      (8 9 10 11 12 13-3)+3
```
1-3⌹ 8 9 10 11 12 13

This inverse relation between addition and subtraction can also be exhibited in terms of maps as follows:

```
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
                                       +3
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
                                       -3
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
                           -3
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
                           +3
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
```
4⌹

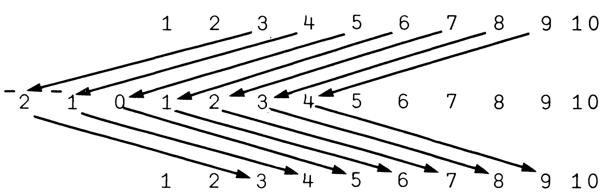
## 3.2 NEGATIVE INTEGERS

Consider a map for the case (3 4 5 6 7 8 9-5)+5 which should yield 3 4 5 6 7 8 9 as a final result:




A problem arises in some of the subtractions, since 3-5 and 4-5 and 5-5 do not yield positive integers. However, the map shows that if we keep track of the unnamed positions to the left of the first positive integer, the overall mapping for adding 5 and then subtracting 5 yields the correct final result.

The problem is resolved by assigning names to each of the new positions as follows:




The first number to the left of 1 is named 0. This is read aloud as "zero", and means "nothing" or "none". The other new numbers, ¯1 and ¯2, are called <u>negative integers</u>, and are read as "negative 1" and "negative 2". Of course, the negative integers continue as far to the left as desired, just as the positive integers continue as far to the right as desired. The whole pattern including the negative integers, zero, and the positive integers, will be called the <u>integers</u>.

The effect of all this is to introduce new integers so that <u>every</u> subtraction has a proper result. Addition and subtraction are still defined as before by moving the proper number of places to the right or left in the pattern of the integers, but the pattern has now been expanded to include the negative integers and zero. 5-6

## 3.3 ADDITION AND SUBTRACTION

The expression 7+¯3 can be considered either as adding 7 to ¯3 as follows:

```
¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7 8

                    +7

¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7 8
```

or as adding ¯3 to 7 as follows:

```
¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7 8

                         +¯3

¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7 8
```

From the above it is clear that adding a negative number is equivalent to subtracting the corresponding positive number; that is, 7+¯3 yields the same result as 7-3. The following examples each show an expression on the left and the corresponding map on the right for a variety of additions and subtractions involving both positive and negative integers:

```
         S←¯2 ¯1 0 1 2

       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
S+3    |                          +3
       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5

       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
S-3    |                          -3
       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5

       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
S+¯3   |                         +¯3
       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5

       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
S-¯3   |                         -¯3
       | ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
```

The first and last examples illustrate that subtraction of a negative number (¯3 in the example) is equivalent to adding the corresponding positive number (3 in the example). This follows from the fact that subtraction of ¯3 is inverse to addition of ¯3 which is equivalent to subtraction of 3. Hence subtraction of ¯3 is inverse to subtraction of 3 and is therefore equivalent to the addition of 3. ▤7-9

## 3.4 EXPRESSIONS FOR PRODUCING THE INTEGERS

The function ⍳ introduced in Chapter 1 produces the positive integers as illustrated below:

```
      ⍳5
1 2 3 4 5
      ⍳7
1 2 3 4 5 6 7
```

The same function can be used to generate both positive and negative integers as follows:

```
      (⍳9)-5
      1 2 3 4 5 6 7 8 9-5
¯4 ¯3 ¯2 ¯1 0 1 2 3 4
      ¯5+⍳9
¯4 ¯3 ¯2 ¯1 0 1 2 3 4
```

The non-negative integers (that is the positive integers and zero), can be generated as follows:

```
      (⍳6)-1
0 1 2 3 4 5
      ¯1+⍳6
0 1 2 3 4 5
```

Non-positive integers can be generated as follows:

```
      ¯8+⍳8
¯7 ¯6 ¯5 ¯4 ¯3 ¯2 ¯1 0
```

The following examples illustrate some functions applied to a vector *S* of integers:

```
      S←¯5+⍳9
      S                                      S+S
¯4 ¯3 ¯2 ¯1 0 1 2 3 4               ¯8 ¯6 ¯4 ¯2 0 2 4 6 8
      1+S                                    2×S
¯3 ¯2 ¯1 0 1 2 3 4 5                ¯8 ¯6 ¯4 ¯2 0 2 4 6 8
      ¯2+S                                   S+S+S
¯6 ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2             ¯12 ¯9 ¯6 ¯3 0 3 6 9 12
      S-S                                    3×S
0 0 0 0 0 0 0 0 0                   ¯12 ¯9 ¯6 ¯3 0 3 6 9 12
```

▤10

# 4

# Function Tables with Negative Integers

## 4.1 INTRODUCTION

Function tables were used in Chapter 2 to explore the behavior of the functions *plus* and *times*. We can now apply them in the same manner to explore the new function *subtraction* introduced in Chapter 3. Moreover, they will be useful in re-examining the behavior of *plus* and *times* when applied to the new negative numbers also defined in Chapter 3.

## 4.2 SUBTRACTION

If `I←⍳9`, then the body of a subtraction table for the arguments 1 to 9 is given by the expression `I∘.-I` as follows:

```
      I←⍳9
      I
1  2  3  4  5  6  7  8  9
      S←I∘.-I
      S
 0 ¯1 ¯2 ¯3 ¯4 ¯5 ¯6 ¯7 ¯8
 1  0 ¯1 ¯2 ¯3 ¯4 ¯5 ¯6 ¯7
 2  1  0 ¯1 ¯2 ¯3 ¯4 ¯5 ¯6
 3  2  1  0 ¯1 ¯2 ¯3 ¯4 ¯5
 4  3  2  1  0 ¯1 ¯2 ¯3 ¯4
 5  4  3  2  1  0 ¯1 ¯2 ¯3
 6  5  4  3  2  1  0 ¯1 ¯2
 7  6  5  4  3  2  1  0 ¯1
 8  7  6  5  4  3  2  1  0
```

The subtraction table `S` has a number of interesting properties. For example, the zeros down the *main diagonal* of the table show that any number subtracted from itself yields 0. Moreover, each diagonal parallel to the main diagonal contains the same number repeated. For example, the diagonal two places below the main diagonal consists entirely of 2's.

Consider the arguments 5 and 3 in the expression 5-3. The result 2 is found in the circled position in the following subtraction table:

| - | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 5 | 4 | 3 | $\boxed{2}$ | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

If each argument is increased by 1, the result is found in the next row and next column; in other words, one place down the diagonal as shown by the square in the above table. Since every entry in this diagonal is the same, we conclude that (5+1) - (3+1) yields the same result as 5-3. More generally, if we increase each argument by any number *N*, the result is found by moving *N* places down the diagonal. Hence we can conclude that (5+*N*) - (3+*N*) yields the same result as 5-3. This conclusion for the arguments 5 and 3 applies to arguments having any values whatever. Hence we conclude that (*X*+*N*) - (*Y*+*N*) yields the same result as *X*-*Y*.

The subtraction table *S* has another interesting property. If we choose the element in the third row and seventh column (which represents the result 3-7), we find that it is the negative of the result in the seventh row and third column (which represents 7-3). Hence the result of 3-7 is the negative of the result of 7-3. If any other pair of numbers is substituted for 7 and 3, the same relation will be observed in the table. We can therefore conclude that for any numbers *X* and *Y*, the result of *X*-*Y* is the negative of the result of *Y*-*X*.

From the above we may conclude the following: if we take the subtraction table *S* and form a new table *T* each of whose <u>columns</u> is equal to the corresponding <u>row</u> of *S*, then each element of *T* will be the negative of the corresponding element of *S*:

*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

*T*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |

The sum of 4 and ¯4 is zero, and in general the sum of any number and its negative is zero. Hence we can state the foregoing result in another way; the sum of the tables *S* and *T* must be a table of all zeros:

*S+T*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| ⌸1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## 4.3 FLIPPING TABLES

In the previous section the table *T* was obtained from the table *S* by interchanging rows and columns. This interchange can be stated in a simple graphic way; flip the table over about the axis formed by the main diagonal:

*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |

In examining the patterns exhibited by tables, it is also convenient to flip them in a similar way about a vertical axis and about a horizontal axis as follows:

*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |

Each of these three methods of flipping a table is a function which takes a table as argument and produces another table as a result. The symbols for each of these functions is a circle with a line through it which indicates the axis about which the table is flipped, thus: ⍉, ⌽, and ⊖. For example:

⍉*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |

⌽*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

⊖*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 |
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 | ¯7 | ¯8 |

⊖⌽*S*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 | 2 |
| ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯8 | ¯7 | ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |

The last of these four examples illustrates how the flipping functions can be applied in succession.

The function ⍉ is called *transposition* (because it transposes rows and columns), the function ⌽ is called *row reversal* (because it reverses each row vector in the table), and ⊖ is called *column reversal*.

A vector can be thought of much as a one-row table, and reversal can therefore be applied to it. For example:

```
      I←⍳9
      I
1  2  3  4  5  6  7  8  9
      ⌽I
9  8  7  6  5  4  3  2  1
```

The relation between the subtraction table *S* and its transpose *T* which was noted at the end of the preceding section can now be stated as follows:

```
      S+⍉S
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0
```

2-3⌹

## 4.4 INDEXING TABLES

In discussing a table it is often necessary to refer to a particular row of the table (e.g., the fourth row), or to a particular column, or to a particular element. Such a reference will be called *indexing* the table, and the row and column numbers which refer to a given element are called its *indices*.

Indexing is denoted by brackets in the manner indicated by the following examples:

```
      M←(⍳6)∘.-⍳6
      M
 0  ¯1  ¯2  ¯3  ¯4  ¯5
 1   0  ¯1  ¯2  ¯3  ¯4
 2   1   0  ¯1  ¯2  ¯3
 3   2   1   0  ¯1  ¯2
 4   3   2   1   0  ¯1
 5   4   3   2   1   0
```

```
      M[3;4]
¯1
      M[4;3]
1
      M[3;]
2 1 0 ¯1 ¯2 ¯3
      M[;3]
¯2 ¯1 0 1 2 3
```

From the first two examples it should be clear that the row index appears first. From the third it appears that a row index alone selects the entire vector in that row. From the fourth it appears that a column index alone selects the entire column. However, the column is displayed horizontally, not as a column. This emphasizes the fact that any single column or row selected from a table is simply a vector and is displayed as such.

Indexing can also be used to select an element from a vector, but in this case a single index only is required:

```
      P←2 3 5 7 11
      P[4]
7
      P[2]
3
      2 3 5 7 11[2]
3
```

Moreover, a vector of indices can be used to select a vector of elements as follows:

```
      P[1 3 5]
2 5 11
      P[⍳4]
2 3 5 7
      P[5 4 3 2 1]
11 7 5 3 2
```

Finally, vectors can be used for both row and column indices to a table as follows:

```
      M[1 2;2 4 6]
¯1 ¯3 ¯5
 0 ¯2 ¯4
      M[1 3;]
 0 ¯1 ¯2 ¯3 ¯4 ¯5
 2  1  0 ¯1 ¯2 ¯3
      M[;2 4 6]
¯1 ¯3 ¯5
 0 ¯2 ¯4
 1 ¯1 ¯3
 2  0 ¯2
 3  1 ¯1
 4  2  0
```

⌸4

## 4.5 ADDITION

Consider the addition table *A* defined as follows:

```
      I←⍳7
      A←I∘.+I
      A
2  3  4  5  6  7  8
3  4  5  6  7  8  9
4  5  6  7  8  9 10
5  6  7  8  9 10 11
6  7  8  9 10 11 12
7  8  9 10 11 12 13
8  9 10 11 12 13 14
```

It is clear that the transpose of the table *A* (that is, ⍉*A*) is equal to *A*. From this we may conclude that for any numbers *X* and *Y*, the sum *X*+*Y* is equal to the sum *Y*+*X*. The diagonals and <u>counter-diagonals</u> (running from upper right to lower left) of the addition table also show interesting patterns whose meanings can be examined in the manner illustrated in the discussion of the subtraction table in Section 4.2.

It is also interesting to examine an addition table made for negative as well as positive arguments as follows:

```
      J←(⍳15)-8
      J
¯7 ¯6 ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7
      B←J∘.+J
      B
¯14 ¯13 ¯12 ¯11 ¯10  ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0
¯13 ¯12 ¯11 ¯10  ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1
¯12 ¯11 ¯10  ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2
¯11 ¯10  ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3
¯10  ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4
 ¯9  ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5
 ¯8  ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5   6
 ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5   6   7
 ¯6  ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5   6   7   8
 ¯5  ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5   6   7   8   9
 ¯4  ¯3  ¯2  ¯1   0   1   2   3   4   5   6   7   8   9  10
 ¯3  ¯2  ¯1   0   1   2   3   4   5   6   7   8   9  10  11
 ¯2  ¯1   0   1   2   3   4   5   6   7   8   9  10  11  12
 ¯1   0   1   2   3   4   5   6   7   8   9  10  11  12  13
  0   1   2   3   4   5   6   7   8   9  10  11  12  13  14
```

One interesting point is that the main diagonal (consisting of all zeros) divides the positive numbers from the negative numbers. Other patterns noted in Table *A* can
5⌹ also be found in the extended Table *B*.

## 4.6 MULTIPLICATION

Again it will be convenient to consider two tables, a multiplication table *M* for positive arguments only, and a multiplication table *N* for negative arguments as well:

```
      I←⍳7
      M←I∘.×I
      M
1   2   3   4   5   6   7
2   4   6   8  10  12  14
3   6   9  12  15  18  21
4   8  12  16  20  24  28
5  10  15  20  25  30  35
6  12  18  24  30  36  42
7  14  21  28  35  42  49

      J←(⍳15)-8
      J
¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1  0  1  2  3  4  5  6  7
      N←J∘.×J
      N
 49  42  35  28  21  14   7  0  ¯7 ¯14 ¯21 ¯28 ¯35 ¯42 ¯49
 42  36  30  24  28  12   6  0  ¯6 ¯12 ¯18 ¯24 ¯30 ¯36 ¯42
 35  30  25  20  15  10   5  0  ¯5 ¯10 ¯15 ¯20 ¯25 ¯30 ¯35
 28  24  20  16  12   8   4  0  ¯4  ¯8 ¯12 ¯16 ¯20 ¯24 ¯28
 21  18  15  12   9   6   3  0  ¯3  ¯6  ¯9 ¯12 ¯15 ¯18 ¯21
 14  12  10   8   6   4   2  0  ¯2  ¯4  ¯6  ¯8 ¯10 ¯12 ¯14
  7   6   5   4   3   2   1  0  ¯1  ¯2  ¯3  ¯4  ¯5  ¯6  ¯7
  0   0   0   0   0   0   0  0   0   0   0   0   0   0   0
 ¯7  ¯6  ¯5  ¯4  ¯3  ¯2  ¯1  0   1   2   3   4   5   6   7
¯14 ¯12 ¯10  ¯8  ¯6  ¯4  ¯2  0   2   4   6   8  10  12  14
¯21 ¯18 ¯15 ¯12  ¯9  ¯6  ¯3  0   3   6   9  12  15  18  21
¯28 ¯24 ¯20 ¯16 ¯12  ¯8  ¯4  0   4   8  12  16  20  24  28
¯35 ¯30 ¯25 ¯20 ¯15 ¯10  ¯5  0   5  10  15  20  25  30  35
¯42 ¯36 ¯30 ¯24 ¯18 ¯12  ¯6  0   6  12  18  24  30  36  42
¯49 ¯42 ¯35 ¯28 ¯21 ¯14  ¯7  0   7  14  21  28  35  42  49
```

The zeros in *N* can be seen to divide the table into four quadrants, one in the upper right corner, one in the upper left, one in the lower left, and one in the lower right. For convenience in referring to them we will call these quadrant 1, quadrant 2, quadrant 3, and quadrant 4, assigning the numbers in a counter-clockwise order beginning with the upper right-hand corner as follows:

```
          quadrant 2          quadrant 1


          quadrant 3          quadrant 4
```

Each quadrant of *N* contains only positive numbers or only negative numbers, and the signs reverse as we proceed counter-clockwise through quadrants 1, 2, 3, and 4. It is also interesting to consider this change of sign by examining some row of the table.

First consider the fourth row of table *M*, which represents the "four times" function for positive arguments:

```
      M[4;]
4  8  12  16  20  24  28
```

Reading this row from left to right is clearly "counting by 4's"; in other words, each entry is obtained from the one before it by adding 4. Similarly, reading backward is equivalent to "counting down by 4's", and each entry is obtained from the one to the right of it by subtracting 4.

Now consider the row of table *N* which represents the same "four times" function, that is, row 12:

```
      N[12;]
¯28  ¯24  ¯20  ¯16  ¯12  ¯8  ¯4  0  4  8  12  16  20  24  28
```

Reading from right to left is again "counting down by fours" and so the entry 4 is preceded by 0 which is in turn preceded by ¯4, and so on. Hence the zero entry separates the positive and negative entries in this row. The same conclusion applies to any row, and a similar conclusion applies to any column. Hence the quadrants must alternate
6-7⊟ in sign, as already observed.

## 4.7 MAXIMUM AND MINIMUM

Consider the following set of positive and negative numbers:

```
      I←(⍳13)-7
      I
¯6  ¯5  ¯4  ¯3  ¯2  ¯1  0  1  2  3  4  5  6
```

For any pair of positive numbers such as 3 and 5, the value of their maximum 3⌈5 is the value of that one of the pair which lies farthest to the right in the vector *I*. The same rule applies to both positive and negative numbers. For example:

```
      3⌈5
5
      3⌈¯5
3
      3⌈I
3  3  3  3  3  3  3  3  3  4  5  6
      ¯3⌈I
¯3  ¯3  ¯3  ¯3  ¯2  ¯1  0  1  2  3  4  5  6
```

Therefore, the maximum table appears as follows:

```
       MAX←I∘.⌈I
       MAX
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5  6
¯5 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5  6
¯4 ¯4 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5  6
¯3 ¯3 ¯3 ¯3 ¯2 ¯1  0  1  2  3  4  5  6
¯2 ¯2 ¯2 ¯2 ¯2 ¯1  0  1  2  3  4  5  6
¯1 ¯1 ¯1 ¯1 ¯1 ¯1  0  1  2  3  4  5  6
 0  0  0  0  0  0  0  1  2  3  4  5  6
 1  1  1  1  1  1  1  1  2  3  4  5  6
 2  2  2  2  2  2  2  2  2  3  4  5  6
 3  3  3  3  3  3  3  3  3  3  4  5  6
 4  4  4  4  4  4  4  4  4  4  4  5  6
 5  5  5  5  5  5  5  5  5  5  5  5  6
 6  6  6  6  6  6  6  6  6  6  6  6  6
```

The corresponding rule for the minimum function is obvious, and the minimum table appears as follows:

```
       MIN←I∘.⌊I
       MIN
¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6 ¯6
¯6 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5 ¯5
¯6 ¯5 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4 ¯4
¯6 ¯5 ¯4 ¯3 ¯3 ¯3 ¯3 ¯3 ¯3 ¯3 ¯3 ¯3 ¯3
¯6 ¯5 ¯4 ¯3 ¯2 ¯2 ¯2 ¯2 ¯2 ¯2 ¯2 ¯2 ¯2
¯6 ¯5 ¯4 ¯3 ¯2 ¯1 ¯1 ¯1 ¯1 ¯1 ¯1 ¯1 ¯1
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  0  0  0  0  0  0
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  1  1  1  1  1
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  2  2  2  2
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  3  3  3
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  4  4
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5  5
¯6 ¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5  6           ⌹8-11
```

## 4.8 RELATIONS

In the work thus far we have observed a number of <u>relations</u> among expressions. For example, 3+8 is equal to 8+3, and in general *X+Y* is equal to *Y+X*. Such relations have also been observed between whole tables. For example, if *M* is any multiplication table it is equal to its transpose ⍉*M*.

The symbol = is used to denote equality, and it will be used as a function which yields a 1 if the arguments are equal, and a 0 if they are not. For example:

```
      3=8
0
      3=3
1
      ¯3=3
0
      I←⍳5
      I
1  2  3  4  5
      ⌽I
5  4  3  2  1
      I=⌽I
0  0  1  0  0

      S←I∘.-I                      M←I∘.×I
      S                            M
  0 ¯1 ¯2 ¯3 ¯4               1   2   3   4   5
  1  0 ¯1 ¯2 ¯3               2   4   6   8  10
  2  1  0 ¯1 ¯2               3   6   9  12  15
  3  2  1  0 ¯1               4   8  12  16  20
  4  3  2  1  0               5  10  15  20  25

      ⍉S                           ⍉M
  0  1  2  3  4               1   2   3   4   5
 ¯1  0  1  2  3               2   4   6   8  10
 ¯2 ¯1  0  1  2               3   6   9  12  15
 ¯3 ¯2 ¯1  0  1               4   8  12  16  20
 ¯4 ¯3 ¯2 ¯1  0               5  10  15  20  25

      S=⍉S                      M=⍉M
 1 0 0 0 0                 1 1 1 1 1
 0 1 0 0 0                 1 1 1 1 1
 0 0 1 0 0                 1 1 1 1 1
 0 0 0 1 0                 1 1 1 1 1
 0 0 0 0 1                 1 1 1 1 1

      S+⍉S                      M-⍉M
 0 0 0 0 0                 0 0 0 0 0
 0 0 0 0 0                 0 0 0 0 0
 0 0 0 0 0                 0 0 0 0 0
 0 0 0 0 0                 0 0 0 0 0
 0 0 0 0 0                 0 0 0 0 0

      0=S+⍉S                    0=M-⍉M
 1 1 1 1 1                 1 1 1 1 1
 1 1 1 1 1                 1 1 1 1 1
 1 1 1 1 1                 1 1 1 1 1
 1 1 1 1 1                 1 1 1 1 1
 1 1 1 1 1                 1 1 1 1 1
```

The symbol ≠ is used to denote the *not-equal* function. For example:

```
      3≠8
1
      3≠3
0
      S≠⍉S
 0 1 1 1 1
 1 0 1 1 1
 1 1 0 1 1
 1 1 1 0 1
 1 1 1 1 0
```

From the foregoing it should be clear that a result of 1 implies that the indicated relation holds (that is, it is *true*), whereas a result of 0 implies that the relation does not hold (that is, it is *false*). ⌹12

There are other useful relations besides equal and not-equal; the symbol < denotes the function *less-than*:

```
      3<5
1
      5<3
0
      3<3
0
      N←(⍳9)-5
      N
¯4  ¯3  ¯2  ¯1  0  1  2  3  4
      ⌽N
4  3  2  1  0  ¯1  ¯2  ¯3  ¯4
      N<⌽N
1  1  1  1  0  0  0  0  0
      (⌽N)<N
0  0  0  0  0  1  1  1  1
```

It should be clear that one integer is less than another if it precedes it in a list of integers (such as *N*) arranged in the usual ascending order.

The symbol > denotes the function *greater-than*. For example:

```
      N>⌽N
0  0  0  0  0  1  1  1  1
      (⌽N)>N
1  1  1  1  0  0  0  0  0
```

To remember which of the symbols < and > denotes less-than and which denotes greater-than, it may be helpful to note that the *large* end of the symbol points to that argument which must be *larger* if the relation is to be true (that is, have the result 1).

Two further relations will also be employed: the *less than or equal to* (denoted by ≤) and the *greater than or equal to* (denoted by ≥). Their definitions should be clear from their names and from the following examples:

```
      I←(⍳7)-4
      I
¯3 ¯2 ¯1  0  1  2  3
      R←⌽I
      R
3  2  1  0 ¯1 ¯2 ¯3
      I≤R
1  1  1  1  0  0  0
      I<R
1  1  1  0  0  0  0
      I=R
0  0  0  1  0  0  0

      I≥R
0  0  0  1  1  1  1
      I>R
0  0  0  0  1  1  1
      I=R
0  0  0  1  0  0  0
```

13-14⌹

## 4.9 LOGICAL VALUES

From all of the examples in the preceding section it can be seen that every result of a relation function is either a 1 or a 0, or a vector or table of 1's and 0's. It will be convenient to use the term *logical* result or *logical* vector or *logical* table to refer to such results which consist of only 0's and 1's. The term "logical" arises from the fact that a 1 can be thought of as representing "true" and a 0 as representing "false".

The functions ⌈ and ⌊ (maximum and minimum) have interesting properties when applied to logical results. The maximum table restricted to such arguments appears as follows:

```
      0 1∘.⌈0 1
0  1
1  1
```

| ⌈ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

From this it appears that the result of *L*⌈*K* (when *L* and *K* are both logical scalars) is 1 if either one of the arguments (or both) is 1. In other words, *L*⌈*K* is *true* if either *L* is true *or* *K* is true. Hence the maximum function applied to logical results can be said to be the function *or*.

The following examples may clarify the matter:

```
      X←1 2 3 4 5
      Y←5 4 3 2 1
      X<Y
1  1  0  0  0
      X=Y
0  0  1  0  0
```

```
      (X<Y)⌈(X=Y)
1  1  1  0  0
      X≤Y
1  1  1  0  0
```

For these values of $X$ and $Y$ it can be seen that the expression $(X<Y)$ ⌈ $(X=Y)$ has the same result as $X \leq Y$. The expression $X<Y)$ ⌈ $(X=Y)$ may be read as "$X$ is less than $Y$ or $X$ equals $Y$" and therefore the conclusion can be phrased as follows: "The expression $X$ is less than $Y$ or $X$ equals $Y$ has the same result as $X \leq Y$".

In a similar manner it can be shown that the minimum function applied to logical results is equivalent to <u>and</u>:

```
      0 1∘.⌊0 1
0  0
0  1
```

| ⌊ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

In other words, the result $L$⌊$K$ is true only if $L$ is true <u>and</u> $K$ is true. For example, $(X \leq Y)$ ⌊ $(X \geq Y)$ is equivalent to $X=Y$. These logical functions are discussed further in Section 14.1.

The function ⌊/$V$ (minimum <u>over</u> $V$) applied to any vector $V$ yields the value of the smallest element in $V$. Hence if $V$ is a logical vector, the expression ⌊/$V$ yields a 0 if there is any zero in $V$, and the expression ⌊/$V$ therefore is true (i.e., 1) only if all elements of $V$ are true. Therefore ⌊/$V$ can be thought of as "all of $V$". Similarly ⌈/$V$ is true if at least one element of $V$ is true. For example:

```
      W←4 6 2 3 7
      1<W
1  1  1  1  1
      ⌊/1<W
1
      ⌈/1<W
1
      3<W
1  1  0  0  1
      ⌊/3<W
0
      ⌈/3<W
1
      8<W
0  0  0  0  0
      ⌊/8<W
0
      ⌈/8<W
0
```

⌹15

## 4.10 THE OVER FUNCTION ON TABLES

The over function has been frequently used on vectors in earlier chapters. For example:

```
      +/2 4 3
9
      ×/2 4 3
24
      ⌈/2 4 3
4
      ⌊/2 4 3
2
```

It is also useful to apply the over function to tables, and the method of doing this will now be defined.

A few examples will be given first:

```
      T←1 2 3 4∘.-1 2 3
      T
 0 ¯1 ¯2
 1  0 ¯1
 2  1  0
 3  2  1
      +/T
¯3 0 3 6
      ×/T
0 0 0 6
      ⌈/T
0 1 2 3
      ⌊/T
¯2 ¯1 0 1
```

The rule should be clear from the foregoing examples: apply the indicated function over each of the vectors formed by the rows of the table.

Sometimes one would like to apply a function over each of the vectors formed by the columns of a table. This can be done by first transposing the table. For example:

```
      ⍉T
 0  1  2  3
¯1  0  1  2
¯2 ¯1  0  1
      +/⍉T
6 2 ¯2
      ×/⍉T
0 0 0
      ⌈/⍉T
3 2 1
      ⌊/⍉T
0 ¯1 ¯2
```

Another over function can of course be applied to any vector resulting from an over function applied to a table. Hence one would obtain the sum of all elements of *T* by the following expression:

```
      +/+/T
6
```

Similarly, the expression ×/+/*T* yields the product of the sums of the rows of *T*:

```
      ×/+/T
0
```

In particular, the expression ⌊/⌊/*L* applied to any logical table *L* will yield a result of 1 (true) only if every element of *L* is true. This is useful in comparing tables. For example:

```
      I←1 2 3 4 5
      S←I∘.-I
      S=⍉S
 1 0 0 0 0
 0 1 0 0 0
 0 0 1 0 0
 0 0 0 1 0
 0 0 0 0 1
      ⌊/⌊/S=⍉S
0
```

```
      A←I∘.+I
      A=⍉A
 1 1 1 1 1
 1 1 1 1 1
 1 1 1 1 1
 1 1 1 1 1
 1 1 1 1 1
      ⌊/⌊/A=⍉A
1
```

⌹16

# 5

# The Rational Numbers

## 5.1 INTRODUCTION

In Chapter 3, the subtraction or minus function was introduced as a function which undid the work of addition, that is, for any positive integers, $X$ and $A$, the expression

$(X+A)-A$

would yield the result $X$. Subtraction was therefore said to be inverse to addition.

Since addition was also inverse to subtraction, it followed that the expression

$(X-A)+A$

would also yield $X$. However, if $A$ is larger than $X$, then $X-A$ is not a positive integer, and the negative integers and zero were introduced to ensure that every subtraction would have a result.

In this chapter the division function will be introduced in a similar way, as a function which will undo the work of multiplication, that is,

$(X\times A)\div A$

yields the result $X$. Since multiplication will also undo the work of division, it follows that

$(X\div A)\times A$

also yields $X$. That is:

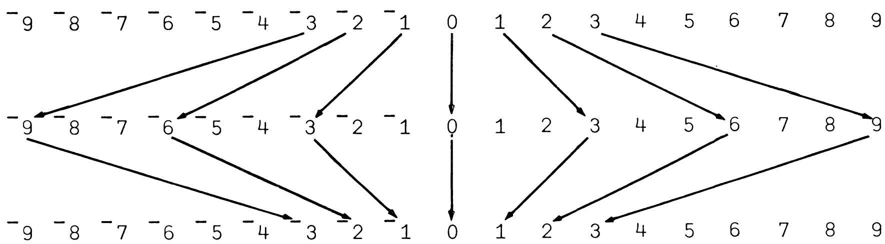
| | READ AS |
|---|---|
| $(X\times A)\div A$ is $X$ | Quantity $X$ times $A$ divided by $A$ is $X$ |
| and | |
| $(X\div A)\times A$ is $X$ | Quantity $X$ divided by $A$ times $A$ is $X$ |

For example:

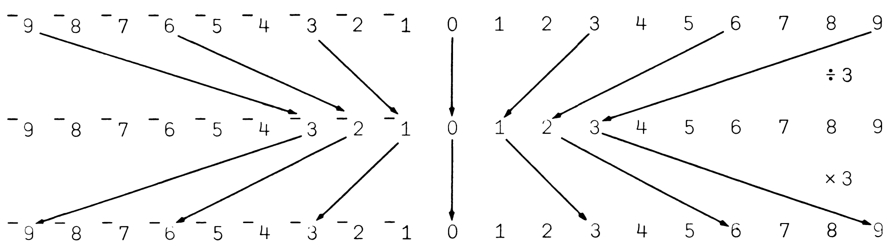
```
      3×8                          3×¯8
24                            ¯24

      (3×8)÷3                      (3×¯8)÷3
8                             ¯8

      24÷3                        ¯24÷3
8                             ¯8

      3×(24÷3)                     3×(¯24÷3)
24                            ¯24

      S←¯4+⍳7
      S
¯3 ¯2 ¯1 0 1 2 3
      S×3
¯9 ¯6 ¯3 0 3 6 9
      (S×3)÷3
¯3 ¯2 ¯1 0 1 2 3
      M←S×3
      M
¯9 ¯6 ¯3 0 3 6 9
      M÷3
¯3 ¯2 ¯1 0 1 2 3
      (M÷3)×3
¯9 ¯6 ¯3 0 3 6 9
```

⌹1-2

Maps for the examples *S*×3 and (*S*×3)÷3 appear as follows:




The examples for *M*÷3 and (*M*÷3)×3 can be mapped similarly:



In discussing the expression $A \div B$, the first argument $A$ is called the dividend (that which is to be divided), the second argument $B$ is called the divisor (that which divides), and the result is called the quotient (how many times). For example, in the expression $12 \div 3$, the number 12 is the dividend, 3 is the divisor, and the result 4 is the quotient.

Just as the expression $X-A$ would sometimes yield a result which was not a positive integer, so the expression $X \div A$ will sometimes yield a result which is not an integer, and it becomes necessary to introduce a new class of numbers which are neither positive nor negative integers. These numbers are called rational numbers because they arise as a ratio of two integers. They are also called fractions, because a number such as $1 \div 3$ is considered to be one piece of a whole which is divided into 3 equal parts, that is, it is a fraction or "fractured part" of a whole. However, the question of these new numbers will be deferred until we have
3⌹ considered methods for performing division.

## 5.2 LONG DIVISION

To divide a small number such as 8 into another small number such as 56, one can simply guess at the answer and then check the guess by multiplying it by the divisor (that is, 8) and comparing the resulting product with the original dividend 56. Thus if the guess is 7, the product $7 \times 8$ is 56 and the guess is correct; the quotient of 56 divided by 8 is 7. More generally, if $DD$ is the name of the dividend, $DR$ is the name of the divisor, and $G$ is the name of the guess, then the product $DR \times G$ must agree with the dividend $DD$ in order that the guess be the correct quotient resulting from $DD \div DR$.

For somewhat larger numbers one is less likely to guess right the first time, and the comparison of the product $DR \times G$ with the dividend $DD$ can be used to determine whether the next guess should be larger or smaller. For example, in the division $40548 \div 124$, the value of $DD$ is 40548, the value of $DR$ is 124, and the first guess $G$ might be slightly over three hundred, say 305. The product of $G$ and $DR$ may then be computed:

```
  124
 ×305
 ----
  620
 000
372
-----
37820
```

Since the product 37820 is less than the dividend 40548, the next guess should be somewhat larger than 305.

One might take the next guess to be 330, in which case the product $124\times330$ would be 40920 and therefore too large. The third guess should be somewhere between 305 (which was too small) and 330 (which was too large). Guessing in this way will eventually lead to the desired quotient, but may take a lot of work. 4

It would help to know not only that the next guess should be larger (or smaller) but by how much. It is easy to find how much the product $DR\times G$ should be increased; one merely subtracts it from the dividend. Thus in the example $40548\div124$ and the guess 305:

```
   124          40548
  ×305         -37820
   620           2728
  000
 372
 37820
```

The product should be increased by 2728. This can be done by increasing the guess by $2728\div124$.

We are thus faced with a new division problem (that is, $2728\div124$), but this time with a smaller dividend. Making a guess of 22 for the quotient would prove correct since $22\times124$ is equal to 2728. The correct quotient is the sum of the first guess (305) and the correction to it (22), that is, 327. The whole process is shown below:

```
  40548÷124

   124        40548        124        2728        305
  ×305       -37820        ×22       -2728        +22
   620         2728        248           0        327
  000                     248
 372                      2728
 37820
```

The work can be organized more conveniently as shown on the left below; the necessary multiplications are shown separately on the right and their results are transferred to the appropriate places on the left:

```
          327
          +22
          305         124          124
  124 ) 40548        ×305          ×22
       -37820         620          248
         2728        000          248
        -2728       372           2728
            0       37820
```

In the foregoing, the final result 327 is entered at the top of the column of guesses (305 and 22) of which it is the sum.

If the second guess is not correct a third can be made, and if that is not correct a fourth can be made, and so on. The final result is the sum of the guesses. For example, to compute 6704÷16:

```
       419                   16          16
        +2                 ×402         ×15
       +15                   32          80
       402                  00           16
   16⟌6704                 64           240
     -6432                 6432
       272                               16
      -240                               ×2
        32                               32
       -32
         0
```

The quotient is 419. This result can be checked by multiplying it by 16 to see that the product is indeed equal
5 to the dividend 6704.

If one chooses each guess to be a single digit, or a single digit followed by one or more zeros (that is, one chooses guesses which are single-digit multiples of 1, 10, 100, 1000, etc.) then the necessary multiplications become much simpler. For example, the division 40548÷124 (used in an earlier example) might begin with a guess of 300. Since 300×124 is equivalent to 3×124 followed by two zeros, this multiplication can be carried out on a single line and need not be done off to the side as was the case with the guess 305 used in the previous example:

```
       300
  124⟌40548
     -37200
       3348
```

The next guess will be a multiple of 10, say 20:

```
        +20
        300
  124⟌40548
     -37200
       3348
      -2480
        868
```

The next guess is a multiple of 1, say 7:

```
           327
           ---
            +7
           +20
           300
      ---------
   124 ) 40548
        -37200
        ------
          3348
         -2480
         -----
           868
          -868
          ----
             0
```

This method of choosing multipliers not only simplifies the necessary multiplications, it also simplifies the addition of the guesses. In the previous example, the addition of 300 and 20 and 7 involves no carries, because each digit position has a single non-zero entry. This will always be the case provided that the leading digit in each guess is chosen as large as possible.

The preceding example (for the division 40548÷124) is repeated below on the left. It is also reproduced on the right but with all of the trailing zeros dropped from the calculations:

```
           327                    327
           ---                    ---
             7                      7
            20                     2
           300                    3
      ---------              ---------
   124 ) 40548            124 ) 40548
        -37200                 -372
        ------                 ----
          3348                  334
         -2480                 -248
         -----                 ----
           868                  868
          -868                 -868
          ----                 ----
             0                    0
```

From this it appears that the simpler scheme on the right will suffice to record the sequence of calculations. In fact, the sequence of guesses 3, 2, and 7 could be written on the same line, making the final addition unnecessary. The steps of this final scheme (called <u>long division</u>) are shown in the columns below:

```
         3                  32                  327
   ---------           ---------           ---------
124 ) 40548        124 ) 40548         124 ) 40548
     -372               -372                -372
     ----               ----                ----
       33                334                 334
                        -248                -248
                        ----                ----
                          86                 868
                                            -868
                                            ----
                                               0
```

⊞6-7

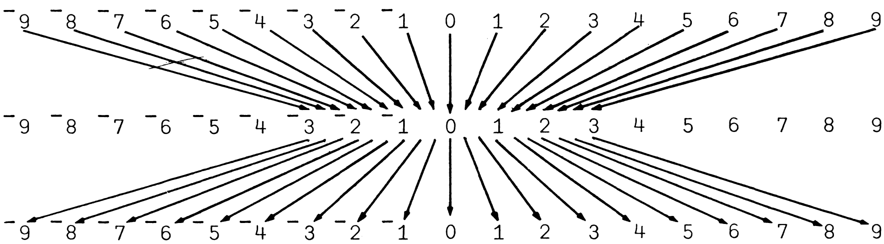
## 5.3 RATIONAL NUMBERS

In the preceding examples and exercises, each dividend used was an integer multiple of the divisor and the quotient was therefore an integer. However, the division 21÷4 cannot have an integer result since the quotient 5 is too small and the quotient 6 is too large. Rational numbers will now be introduced to ensure that a quotient such as 21÷4 has a result.

Consider the example

```
      P←¯10+⍳19
      P
¯9 ¯8 ¯7 ¯6 ¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5 6 7 8 9
```

and the following map for *P*÷3 and (*P*÷3)×3




From this example, it appears that the number 6÷3 is less than 7÷3 which is less than 8÷3, and so on. In other words, the following sequence of four numbers is in ascending order:

```
6÷3    7÷3    8÷3    9÷3
```

Since 6÷3 is 2 and 9÷3 is 3, the above may be written as:

```
2    7÷3    8÷3    3
```

In other words, the numbers 7÷3 and 8÷3 occur between the integers 2 and 3 and therefore cannot be integers. They are called <u>rational numbers</u>.

The negative integers and zero (introduced to make every subtraction have a result) are a set of numbers which <u>precede</u> the positive integers; the rational numbers (introduced to make every division have a result) are a set of numbers which occur <u>between</u> the integers.

Just as names were introduced for the negative numbers (for example ¯5 ¯4 ¯3), names can be introduced for rationals. The result of 2÷3 is often written as 2/3, the result of 5÷2 is written as 5/2, etc. In this book we will make very little use of such names, but will instead simply write the expression which produces the rational number (for example, 2÷3 or 5÷2, or ÷/2 3 or ÷/5 2), or else write the rational number as a decimal fraction. Decimal fractions will be discussed later in this chapter.

Since the integer 2 is equal to 2÷1 or to 4÷2 or to 6÷3, etc., then the integer 2 itself can be considered to be a rational number. Similarly, 3 is equal to 3÷1 or 6÷2, etc. Therefore every integer can also be considered to be a rational number.

In discussing a rational such as *A*÷*B*, the terms dividend and divisor were introduced to refer to the parts *A* and *B*. The terms numerator (for *A*) and denominator (for *B*) are also used. To denominate means "to give a name to", and the second part of a rational gives a name to the result in the following sense: 3÷5 is called 3 fifths, 5÷7 is called 5 sevenths, etc. Similarly, the numerator gives the number of things named, as also illustrated in the examples of the preceding sentence. ▣8-11

## 5.4 ADDITION OF RATIONALS HAVING THE SAME DIVISOR

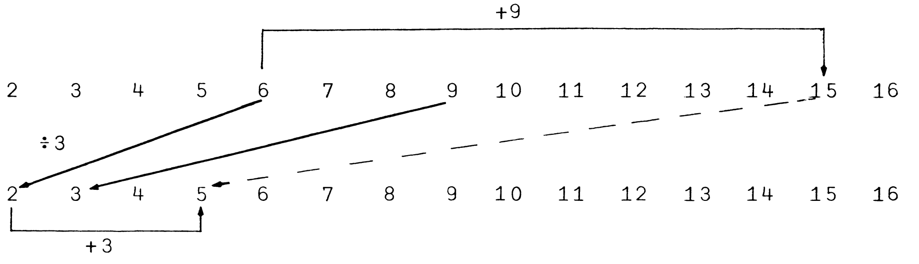
Consider the following pairs of examples:

| | |
|---|---|
| (6÷3)+(9÷3) | (6+9)÷3 |
| 5 | 5 |
| (20÷5)+(25÷5) | (20+25)÷5 |
| 9 | 9 |
| (32÷4)+(8÷4) | (32+8)÷4 |
| 10 | 10 |

Since each of the results in the first column agrees with the corresponding result in the second column, it appears that the expressions in each pair are equivalent, that is, (9÷3)+(6÷3) is equivalent to (9+6)÷3, and so forth. The general rule illustrated by the examples is this: If *A*, *B*, and *C* are any three integers, then

(*A*÷*C*)+(*B*÷*C*) is equal to (*A*+*B*)÷*C*

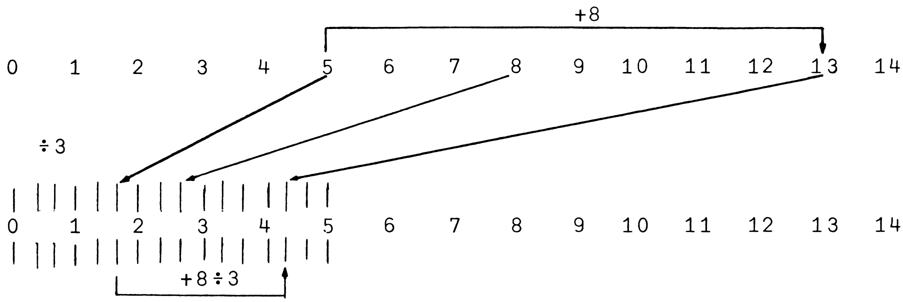
The first example may be diagrammed as follows:



Each division in the foregoing examples produces an integer, and so the rule for addition deduced above has only been shown to hold for such cases. It will, however, be assumed to hold for all rational numbers. For example:

(5÷3)+(8÷3) is equal to 13÷3

The diagram for this example follows:




It should be clear from the foregoing that similar rules apply to the subtraction of rationals having the same divisor, that is:

($A$÷$C$)-($B$÷$C$) is equal to ($A$-$B$)÷$C$

For example:

(13÷3)-(8÷3) is equal to 5÷3.

If the addition or subtraction of two rationals produces a dividend which is evenly divisible by the divisor, then the result may be further simplified to a single integer. For example:

| (8÷3)+(7÷3)
| 15÷3
| 5

| (8÷3)-(5÷3)
| 3÷3
| 1

The vertical lines above indicate, as usual, that the expressions to the right are equivalent. From here on the vertical lines will be omitted; that is, any list of expressions should be read as a statement that the
12-14⊞ expressions are equivalent.

## 5.5 MULTIPLICATION OF RATIONAL NUMBERS

The rules for multiplying two rational numbers will be explored by first considering a number of cases in which the division can actually be performed. Compare the corresponding examples in the following two columns:

| | |
|---|---|
| $(10\div5)\times(12\div3)$ | $(10\times12)\div(5\times3)$ |
| $2\times4$ | $120\div15$ |
| $8$ | $8$ |
| $(18\div3)\times(12\div6)$ | $(18\times12)\div(3\times6)$ |
| $6\times2$ | $216\div18$ |
| $12$ | $12$ |
| $(32\div8)\times(35\div7)$ | $(32\times35)\div(8\times7)$ |
| $4\times5$ | $1120\div56$ |
| $20$ | $20$ |

Since the results in the two columns agree, it appears that $(10\div5)\times(12\div3)$ is equivalent to $(10\times12)\div(5\times3)$ and so on. In general, if $A$, $B$, $C$, and $D$ are any integers, it appears that $(A\div B)\times(C\div D)$ is equivalent to $(A\times C)\div(B\times D)$. The above examples illustrate this only for cases where $A\div B$ and $C\div D$ each produce integer results. However, the rule will be assumed to apply for all rational numbers. For example:

$(3\div4)\times(5\div2)$ is equal to $15\div8$

$(4\div3)\times(2\div5)$ is equal to $8\div15$

$(3\div4)\times(4\div3)$ is equal to $12\div12$ (that is, $1$).

The rule for multiplying rationals can therefore be stated as follows:

$(A\div B)\times(C\div D)$

$(A\times C)\div(B\times D)$

In words, the dividend of the result is the product of the dividends and the divisor of the result is the product of the divisors.

Applying this rule to the case where $A$, $B$, $C$, and $D$ are equal to $4$, $5$, $3$, and $3$, respectively, yields

$(4\div5)\times(3\div3)$
$(4\times3)\div(5\times3)$
$12\div15$

However, since $3 \div 3$ is $1$, then

$(4 \div 5)\times(3 \div 3)$
$(4 \div 5)\times 1$
$4 \div 5$

Therefore, all members of the two sets of expressions above are equivalent, and $12 \div 15$ is equal to $4 \div 5$.

It therefore appears that for any three integers $A$, $B$, and $C$:

$A \div B$
$(A \div B)\times(C \div C)$
$(A \times C)\div(B \times C)$

In words, if the dividend and divisor of a rational number are multiplied by the same quantity $C$, the resulting
15-18⌹ rational number is equal to the original rational number.

## 5.6 MULTIPLICATION OF A RATIONAL BY AN INTEGER

Consider again the general rule for the multiplication of two ratios, that is:

$(A \div B)\times(C \div D)$
$(A \times C)\div(B \times D)$

If $B$ has the value $1$, we obtain the following simpler rule:

$A \times (C \div D)$
$(A \div 1)\times(C \div D)$
$(A \times C)\div(1 \times D)$
$(A \times C)\div D$

In other words, if a ratio $C \div D$ is to be multiplied by an integer $A$, the result is obtained by simply multiplying the numerator $C$ by $A$. For example:

$5 \times (3 \div 7)$
19⌹ $15 \div 7$

## 5.7 MULTIPLICATION EXPRESSED IN TERMS OF VECTORS

Since $3 \div 4$ can be written as ÷/3 4, and $5 \div 2$ can be written as ÷/5 2, etc., then any rational can be written as ÷/*V*, where *V* is a two-element vector. The first examples used in the multiplication of rational numbers will now be repeated but written in this new form:

```
      (÷/10 5)×(÷/12 3)
      2×4
8
```

```
      ÷/10 5×12 3
      ÷/120 15
8
```

```
      (÷/18 3)×(÷/12 6)
      6×2
12
```

```
      ÷/18 3×12 6
      ÷/216 18
12
```

```
      (÷/32 8)×(÷/35 7)
      4×5
20
```

```
      ÷/32 8×35 7
      ÷/1120 56
20
```

From the foregoing it appears that the rule for multiplying rationals can be written very neatly in terms of vectors: if *V* and *W* are each two-element vectors, then the product of the rationals (÷/*V*)×(÷/*W*) is equivalent to the rational ÷/*V*×*W*. For example:

```
      V←10 5
      W←12 3
      (÷/V)×(÷/W)
      2×4
8
      V×W
120 15
      ÷/V×W
8
```

⌹20

## 5.8 ADDITION OF RATIONALS

The method for adding rationals given in Section 5.4 applied only to the addition of two rationals sharing the same divisor, that is,

(*A*÷*C*)+(*B*÷*C*) is equal to (*A*+*B*)÷*C*

It cannot be applied to add a pair of rationals such as 2÷3 and 4÷5. However, the results of the preceding section can be applied as follows:

2÷3 is equal to (2×5)÷(3×5)

4÷5 is equal to (4×3)÷(5×3)

Therefore 2÷3 and 4÷5 are equal to 10÷15 and 12÷15, respectively. But the last two rationals have the same divisor and can therefore be added as follows:

(10÷15)+(12÷15) is equal to 22÷15.

Therefore

(2÷3)+(4÷5) is equal to 22÷15.

Similarly:

```
(2÷7)+(4÷5)
((2÷7)×(5÷5))+((4÷5)×(7÷7))
(10÷35)+(28÷35)
38÷35

(1÷2)+(1÷3)+(1÷6)
((1÷2)×(3÷3))+((1÷3)×(2÷2))+(1÷6)
(3÷6)+(2÷6)+(1÷6)
6÷6
1
```

In general, two rationals, (*A*÷*B*) and (*C*÷*D*) may be added as follows:

```
    (A÷B)+(C÷D)
    ((A÷B)×(D÷D))+((C÷D)×(B÷B))
    ((A×D)÷(B×D))+((C×B)÷(D×B))
21  ((A×D)+(C×B))÷(B×D)
```

## 5.9 ADDITION OF RATIONALS IN TERMS OF VECTORS

Recall the rule for the addition of two rationals as follows:

```
(A÷B)+(C÷D)
((A×D)+(B×C))÷(B×D)
```

Recall also that if *V* is a two element vector, then ÷/*V* is the ratio *V*[1]÷*V*[2]. Consequently, the rule for the addition of two rationals ÷/*V* and ÷/*W* can be expressed as follows:

```
(÷/V)+(÷/W)
(+/V×⌽W)÷(V[2]×W[2])
```

For example:

```
    V←3 5
    W←7 2
    (÷/3 5)+(÷/7 2)
    (+/3 5×2 7)÷(5×2)
    (+/6 35)÷10
22  41÷10
```

## 5.10 THE QUOTIENT OF TWO RATIONALS

Consider the following examples of division:

```
      12÷4
3
```

$(12\times5)\div(4\times5)$

3

$18\div2$

9

$(18\times7)\div2\times7$

9

They illustrate the fact, developed earlier, that the multiplication of both numerator and denominator by the same quantity leaves a fraction unchanged. That is:

$P\div Q$
$(P\times R)\div(Q\times R)$

Consider now the division of the rational number $A\div B$ by the rational number $C\div D$, that is,

$(A\div B)\div(C\div D)$

The result will remain unchanged if the numerator $A\div B$ and the denominator $C\div D$ are each multiplied by the same number $D\div C$. That is:

$(A\div B)\div(C\div D)$
$((A\div B)\times(D\div C))\div((C\div D)\times(D\div C))$

The last half of the above expression (that is, $(C\div D)\times(D\div C)$) can be simplified by applying the rule that the product of two rationals is the product of their numerators divided by the product of their denominators:

$(C\div D)\times(D\div C)$
$(C\times D)\div(D\times C)$

Since $C\times D$ and $D\times C$ are equal, their quotient is 1. Therefore $(C\div D)\times(D\div C)$ makes 1.

Finally, then:

$(A\div B)\div(C\div D)$
$((A\div B)\times(D\div C))\div((C\div D)\times(D\div C))$
$((A\div B)\times(D\div C)\div1$
$(A\div B)\times(D\div C)$

Therefore the quotient $(A\div B)\div(C\div D)$ is equivalent to the product $(A\div B)\times(D\div C)$. For example:

$(36\div3)\div(24\div4)$

2

$(36\div3)\times(4\div24)$

2

This relation can also be expressed in terms of vectors as follows. If *V* is a two-element vector and *W* is a two-element vector, then:

```
      (÷/V)÷(÷/W)
      (÷/V)×÷/⌽W
```

For example:

```
      (÷/36 3)÷(÷/24 4)
2

      (÷/36 3)×(÷/4 24)
2
```

23⍞

## 5.11 DECIMAL FRACTIONS

Any rational number having a denominator such as 10 or 100 or 1000, etc., can be represented as a *decimal fraction* in the manner illustrated below:

```
      1386÷10
      138.6

      1386÷100
      13.86

      1386÷1000
      1.386

      1386÷10000
      .1386

      1386÷100000
      .01386
```

The period occurring in a decimal fraction is called a *decimal point*. If the decimal point in a decimal fraction is followed by one digit, then the rational it represents is the integer represented by the same digits without a decimal point, divided by 10. If the decimal point is followed by two digits, the rational represented is the same integer divided by 100, and, in general, if the decimal point is followed by *K* digits, then the rational represented is the same integer divided by the integer formed by a 1 followed
24-26⍞ by *K* zeros.

## 5.12 ADDITION AND SUBTRACTION OF DECIMAL FRACTIONS

The following examples show the addition of some pairs of decimal fractions in which the fractions in each pair have the decimal point <u>in the same place</u>, that is, they have the same number of digits following the decimal place:

$21.34+16.55$
$(2134\div100)+(1655\div100)$
$(2134+1655)\div100$
$3789\div100$
$37.89$

$13.659+82.546$
$(13659+82546)\div1000$
$96205\div1000$
$96.205$

$12.700+39.615$
$(12700+39615)\div1000$
$52.315$

In other words, a pair of decimal fractions having the decimal point in the same place can be added just as if they were integers (i.e., by ignoring the decimal point), and then placing the decimal point in the same place in the result. This rule may be applied to the foregoing examples as follows:

$$\begin{array}{r} 21.34 \\ \underline{16.55} \\ 37.89 \end{array} \qquad \begin{array}{r} 13.659 \\ \underline{82.546} \\ 96.205 \end{array} \qquad \begin{array}{r} 12.700 \\ \underline{39.615} \\ 52.315 \end{array}$$

By the same reasoning, subtraction of such a pair of decimal fractions can be carried out in a similar manner. For example, the subtraction $21.34-16.55$ can be carried out as follows:

$$\begin{array}{r} 21.34 \\ \underline{16.55} \\ 4.79 \end{array}$$ 27

It remains to add two decimal fractions which do not have the same number of digits following the decimal point. The value of a decimal fraction is not changed by appending zeros to the right of it; thus $12.7$ and $12.70$ and $12.700$, etc., are all equal. This follows from the fact (established earlier) that the value of a rational is unchanged if the numerator and denominator are each multiplied by the same number. For example:

12.7
127÷10
(127×10)÷(10×10)
1270÷100
12.70
1270÷100
(1270×10)÷(100×10)
12700÷1000
12.700

Therefore, zeros may be appended to the right of any decimal fraction without changing its value. To perform the addition 12.7+39.615, one appends two zeros to the right of 12.7 (getting 12.700) and then adds them by the method for adding decimal fractions having the decimal point in the same place:

12.700
39.615
28 52.315

## 5.13 THE DECIMAL FRACTION REPRESENTATION OF A RATIONAL

Many rational numbers having denominators which are not of the form 10, 100, 1000, etc., can still be expressed as decimal fractions by simply multiplying both numerator and denominator by some integer which produces a denominator which is of the form 10, 100, 1000, etc. For example:

1÷2
(1×5)÷(2×5)
5÷10
.5

3÷5
6÷10
.6

7÷2
35÷10
3.5

1÷25
4÷100
.04

38÷4
950÷100
9.5

1÷125
8÷1000
.008

1÷16
625÷10000
.0625

1÷625
16÷10000
.0016

From these examples, it should be clear that the ordinary long division process may be used to convert such rationals to decimal fractions; all that is needed is to append to the integer numerator a decimal point followed by a sufficient number of zeros. For example, since 38 is equivalent to 38.0 then 38÷4 may be written as 38.0÷4 and the long division may be carried out as follows:

```
     9.5
  4 |38.0
   -36
     20
    -20
      0
```

Similarly, ÷/1 16 may be converted to decimal fraction as follows:

```
      .0625
  16 |1.0000
      -96
        40
       -32
        80
       -80
         0
```

29

## 5.14 DECIMAL FRACTION APPROXIMATIONS TO RATIONALS

The rational number 75÷64 can be converted to a decimal fraction by long division as follows:

```
      1.171875
  64 |75.000000
     -64
      110
      -64
       460
      -448
        120
        -64
         560
        -512
          480
         -448
           320
          -320
             0
```

Therefore, 75÷64 is equivalent to 1.171875.

Suppose that one stopped the long division process just before the last digit, obtaining the quotient 1.17187 and leaving a non-zero remainder, that is, 320. The decimal fraction 1.17187 is <u>not</u> equal to 75÷64, but it <u>is</u> very nearly equal to it and is therefore said to be a good <u>approximation</u> to 75÷64. To see how close 1.17187 is to

75÷64 one may subtract the approximation 1.17187 from the true value 1.171875 as follows:

```
  1.171875
 -1.171870
  0.000005
```

The difference is therefore .000005 or 5÷1000000. This is only 5 millionths, a very small quantity.

The decimal fraction 1.17187 is said to be a 5-place approximation to 75÷64 because it is close to 75÷64 and has 5 digits following the decimal place. It is also a best 5-place approximation to 75÷64, since no other decimal fraction with only 5 places can be closer (although 1.17188 is just as close and is also a best approximation).

The decimal fraction 1.171 (obtained by stopping the long division after 3 places) is a three-place approximation to 75÷64, and is smaller than 75÷64 by the amount .000875. It is not, however, the best approximation, since the fraction 1.172 is larger than 75÷64 by only .000125 as may be seen from the following subtraction:

```
  1.172000
 -1.171875
  0.000125
```

Therefore, to get a best approximation to a rational, one should continue the long division one place beyond the desired number of places. If the additional digit is less than 5, the additional digit should be discarded; if not, the additional digit should be discarded but a 1 should be added into the last place kept. For example:

```
       1.1718
  64 ) 75.0000
      -64
       110
       -64
        460
       -448
         120
         -64
          560
         -512
           48
```

The best three-place approximation is 1.171+.001, or 1.172.

Similarly, the best two-place approximation to 115÷64 can be obtained as follows:

```
      1.796
 64 ⌐115.000
     -64
      510
     -448
       620
      -576
        440
       -384
         56
```

The best two-place approximation to 115÷64 is therefore 1.79+.01, which is 1.80, or simply 1.8.

For many rationals, the long division process <u>never</u> terminates with a zero remainder. For example, for the rational 1÷3, the remainder is always 1:

```
     .333
  3 ⌐1.000
     -9
      10
      -9
       10
       -9
        1
```

For such a case, the long division process can also be used to give a best approximation to the rational, thus .333 is the best 3-place approximation for the rational 1÷3 and differs from it by only 1÷3000. For,

```
.333+(1÷3000)
(333÷1000)+(1÷3000)
(999÷3000)+(1÷3000)
1000÷3000
1÷3
```

Similarly, .667 may be obtained as the best 3-place approximation to 2÷3 as follows:

```
     .6666
  3 ⌐2.0000
   -1 8
      20
     -18
       20
      -18
        20
       -18
         2
```

Since the fourth digit of the result exceeds 5, the best three-place approximation is .666+.001, or .667.

The following table shows the five-place decimal fraction approximations to the rationals resulting from the expression (⍳7)∘.÷⍳7:

```
    1.00000  0.50000  0.33333  0.25000  0.20000  0.16667  0.14286
    2.00000  1.00000  0.66667  0.50000  0.40000  0.33333  0.28571
    3.00000  1.50000  1.00000  0.75000  0.60000  0.50000  0.42857
    4.00000  2.00000  1.33333  1.00000  0.80000  0.66667  0.57143
    5.00000  2.50000  1.66667  1.25000  1.00000  0.83333  0.71429
    6.00000  3.00000  2.00000  1.50000  1.20000  1.00000  0.85714
30⌹ 7.00000  3.50000  2.33333  1.75000  1.40000  1.16667  1.00000
```

## 5.15 MULTIPLICATION OF DECIMAL FRACTIONS

The following example shows the multiplication of two decimal fractions:

```
      1.3×2.14
      (13÷10)×(214÷100)
      (13×214)÷(1000)
      2782÷1000
2.782
```

From this it is clear that the following rule can be used: multiply the numbers as integers (ignoring the decimal point) and place a decimal point in the result so that the number of digits following it is equal to the sum of the number of digits following the decimal points in the two factors. For example:

```
           2.14        (2 decimal places)
            1.3        (1 decimal place)
           ----
            642
           214
           ----
31-32⌹     2.782       (2+1 decimal places)
```

## 5.16 DIVISION OF DECIMAL FRACTIONS

The following procedure can be used to find the quotient where the dividend and divisor are decimal fractions:

1. Perform the division as if the numbers were integers, ignoring the decimal points.

2. In the resulting quotient, move the decimal point as many places to the left as there are decimal places in the original dividend.

3. From there move the decimal point as many places to the right as there are decimal places in the original divisor.

For example, to evaluate the expression 11.025÷1.26, we first divide the integer 11025 by the integer 126:

```
        87.5
 126⌐11025
    -1008
       945
      -882
       630
      -630
         0
```

The decimal point in the quotient 87.5 is now moved three places to the left (because the dividend 11.025 has three decimal places) to obtain .0875, and the decimal place is then moved 2 places to the right (because the divisor 1.26 has two decimal places) to obtain 8.75. This result can be checked by evaluating 8.75×1.26 to see that it yields 11.025 as required.

The justification for this procedure should be clear from the following equivalences:

```
11.025÷1.26
(11025÷1000)÷(126÷100)
(11025÷1000)×(100÷126)
(11025÷126)×(100÷1000)
```

⌹33

## 5.17 EXPONENTIAL NOTATION

Numbers such as 120000000 and .0000000017 are awkward to read and write because of the large number of zeros to be counted. Exponential notation allows one to write these numbers instead as 12*E*7 and 17*E*¯10.

More generally, one may write any decimal number (or integer) followed immediately by an *E* followed immediately by an integer. The value this denotes may be determined as follows: take the number before the *E* and move its decimal point by an amount determined by the integer following the *E*, moving it to the right if the integer is positive and to the left if the integer is negative. For example:

```
1.34E5          1.34E¯5
134000          .0000134
134E3           134E¯7
.134E6          .134E¯4
```

⌹34-35

## 5.18 DIVISION WITH NEGATIVE ARGUMENTS

A study of the map used in introducing rational numbers (Section 5.1) should make it clear that (¯1)÷3 is the negative of 1÷3, that (¯2)÷3 is the negative of 2÷3, etc. The result to be obtained when the divisor is negative is not so clear.

Consider the rational 3÷¯4 which has a negative divisor. We have seen that it is equivalent to the rational (3×*A*)÷(¯4×*A*), where *A* is any integer. If we choose *A* to be ¯1, then (3×*A*)÷(¯4×*A*) is equal (¯3)÷4. Similarly, (¯3)÷(¯4) is equal to 3÷4. From this it appears that the sign of the quotient *B*÷*C* is determined from the signs of the arguments *B* and *C* in exactly the same way that the sign of the product *B*×*C* is determined (as illustrated by the large table in
36⍈ Section 4.6).

## 5.19 DIVISION BY ZERO

The result of the division *A*÷*B* is a quotient *C* such that *C*×*B* is equal to *A*. If *A* is 4 and *B* is zero, then *C* must be a number such that *C*×0 is 4. Since 0 times anything is 0, there is no such number *C*. Hence division by zero is not possible.

# 6

# Function Tables with Rational Numbers

## 6.1 INTRODUCTION

In Chapter 4 we used function tables to examine the function of subtraction newly introduced in Chapter 3, and to re-examine familiar functions applied to the negative numbers also introduced in Chapter 3. In this chapter we will pursue a similar course with respect to the division function and the rational numbers introduced in Chapter 5.

In this chapter, the results of divisions are represented as decimal fractions correct to three places.

## 6.2 CATENATION

Catenation is a simple new function which will be needed in this and later chapters; it is denoted by the comma. "Catena" is a Latin word meaning "chain", and *catenation* is a function which chains its arguments together. For example:

```
      X←1 2 3
      Y←4 5
      X,Y
1 2 3 4 5
      Y,X
4 5 1 2 3
      +/X,Y
15
      X,7
1 2 3 7
      7,X
7 1 2 3
      7,8
7 8
```

## 6.3 DIVISION TABLES

If *I*←⍳8, then the body of the division table for the arguments 1 to 8 is given by the expression *I*∘.÷*I* as follows:

```
      I←⍳8
      D←I∘.÷I
      D
 1.000  0.500  0.333  0.250  0.200  0.167  0.143  0.125
 2.000  1.000  0.667  0.500  0.400  0.333  0.286  0.250
 3.000  1.500  1.000  0.750  0.600  0.500  0.429  0.375
 4.000  2.000  1.333  1.000  0.800  0.667  0.571  0.500
 5.000  2.500  1.667  1.250  1.000  0.833  0.714  0.625
 6.000  3.000  2.000  1.500  1.200  1.000  0.857  0.750
 7.000  3.500  2.333  1.750  1.400  1.167  1.000  0.875
 8.000  4.000  2.667  2.000  1.600  1.333  1.143  1.000
```

This table has a number of interesting properties. For example, each row can be seen to be in descending order and each column can be seen to be in ascending order. Moveover, the main diagonal consists of all 1's, illustrating the fact that *N*÷*N* is equal to 1 whatever the value of *N*. Moreover, many other duplications occur in the table, showing that the same value may result from the division of different pairs of numbers. Thus the decimal fraction 0.333 occurs in two places, resulting from 1÷3 and
2⌹ 2÷6.

The division table can be extended to negative arguments as well. However, as pointed out in Chapter 5, the number 0 is not permitted as the right argument of division:

```
      J←(⍳9)-5
      J
¯4  ¯3  ¯2  ¯1  0  1  2  3  4
      K←(0-⌽⍳4),⍳4
      K
¯4  ¯3  ¯2  ¯1  1  2  3  4
      J∘.÷K
 1.000   1.333   2.000   4.000  ¯4.000  ¯2.000  ¯1.333  ¯1.000
 0.750   1.000   1.500   3.000  ¯3.000  ¯1.500  ¯1.000  ¯0.750
 0.500   0.667   1.000   2.000  ¯2.000  ¯1.000  ¯0.667  ¯0.500
 0.250   0.333   0.500   1.000  ¯1.000  ¯0.500  ¯0.333  ¯0.250
 0.000   0.000   0.000   0.000   0.000   0.000   0.000   0.000
¯0.250  ¯0.333  ¯0.500  ¯1.000   1.000   0.500   0.333   0.250
¯0.500  ¯0.667  ¯1.000  ¯2.000   2.000   1.000   0.667   0.500
¯0.750  ¯1.000  ¯1.500  ¯3.000   3.000   1.500   1.000   0.750
¯1.000  ¯1.333  ¯2.000  ¯4.000   4.000   2.000   1.333   1.000
```

3⌹

## 6.4 COMPARISON

Two rationals such as `3÷7` and `4÷9` can be compared to see which is the larger by first converting them each to a decimal representation. For example:

```
      3÷7
0.429
      4÷9
0.444
      (3÷7)≤(4÷9)
1
```

It is also possible to compare two rationals without actually carrying out any division.

If two rationals have the same denominator, they can be compared by simply comparing their numerators. For example, `27÷63` is less then `28÷63`. Moreover, for any pair of fractions one can find an equivalent pair which do have the same denominator. For example, `3÷7` is equivalent to `(3×9)÷(7×9)` (that is, `27÷63`) and `4÷9` is equivalent to `(7×4)÷(7×9)` (that is, `28÷63`).

In general, if `N1`, `D1`, `N2`, and `D2` are any integers, then `N1÷D1` and `N2÷D2` can be compared by forming the equivalent pair `(N1×D2)÷(D1×D2)` and `(D1×N2)÷(D1×D2)`, which have the same denominator. Hence it is only necessary to compare the numerators `N1×D2` and `D1×N2`. For example:

```
      N1←3
      D1←7
      N2←4
      D2←9
      N1÷D1
0.429
      N2÷D2
0.444
      (N1÷D1)≤(N2÷D2)
1
      (N1×D2)≤(D1×N2)
1
```

The same relations will of course hold if `N1`, `D1`, `D2`, and `D2` are vectors. For example:

```
      N1←1 1 1 2 2 2 3 3 3
      D1←1 2 3 1 2 3 1 2 3
      N2←4 4 4 5 5 5 6 6 6
      D2←4 5 6 4 5 6 4 5 6

      N1÷D1
1  0.5  0.333  2  1  0.667  3  1.5  1
      N2÷D2
1  0.8  0.667  1.25  1  0.833  1.5  1.2  1
```

```
      (N1÷D1)≤(N2÷D2)
1  1  1  0  1  1  0  0  1
      (N1×D2)≤(D1×N2)
1  1  1  0  1  1  0  0  1
```

Moreover, if one wants to compare each element of *N1÷D1* with each element of *N2÷D2*, then the corresponding comparison tables agree as well:

```
      (N1÷D1)∘.≤(N2÷D2)          (N1∘.×D2)≤(D1∘.×N2)
 1 0 0 1 1 0 1 1 1          1 0 0 1 1 0 1 1 1
 1 1 1 1 1 1 1 1 1          1 1 1 1 1 1 1 1 1
 1 1 1 1 1 1 1 1 1          1 1 1 1 1 1 1 1 1
 0 0 0 0 0 0 0 0 0          0 0 0 0 0 0 0 0 0
 1 0 0 1 1 0 1 1 1          1 0 0 1 1 0 1 1 1
 1 1 1 1 1 1 1 1 1          1 1 1 1 1 1 1 1 1
 0 0 0 0 0 0 0 0 0          0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 1 0 0          0 0 0 0 0 0 1 0 0
 1 0 0 1 1 0 1 1 1          1 0 0 1 1 0 1 1 1

      ⌊/⌊/((N1÷D1)∘.≤(N2÷D2)) = ((N1∘.×D2)≤(D1∘.×N2))
4⌹ 1
```

## 6.5 THE POWER FUNCTION FOR NEGATIVE AND ZERO ARGUMENTS

In Chapter 4 the functions +, ×, ⌈, and ⌊ were re-examined to determine how they applied to the negative arguments introduced in Chapter 3. This was not done for the power function because the result of an expression such as 2*¯3 is a rational number, and rational numbers had not yet been introduced.

We will begin by recalling the definition of the power function as the product over a number of repetitions of a certain factor, that is, *A*B* is equivalent to ×/*B*⍴*A*. For example:

```
      3⍴2
2  2  2
      ×/3⍴2
8
      2*3
8
```

The power table for positive integers therefore appears as follows:

```
      I←2 3 4 5 6
      J←2 3 4 5 6 7
      I∘.*J
 4       8      16      32      64     128
 9      27      81     243     729    2187
16      64     256    1024    4096   16384
25     125     625    3125   15625   78125
36     216    1296    7776   46656  279936
```

A simple pattern emerges in each row of the table: any element of a row can be obtained from the element which precedes it by multiplying by a certain factor, that factor being the value of the left argument which produced that row. For example, the third row was produced by the expression:

```
      4*2 3 4 5 6 7
16  64  256  1024  4096  16384
```

and the third element in the row can be obtained from the one before it by multiplying by 4.

This same pattern can be stated in a different way: each element can be obtained from the one following it by dividing by the same factor. In this way the pattern can be extended to the left to obtain results for right arguments less than 2:

```
      I←2 3 4 5
      J←(⍳7)-4
      J
¯3  ¯2  ¯1  0  1  2  3
      I∘.*J
 0.125    0.250    0.500    1.000    2.000    4.000    8.000
 0.037    0.111    0.333    1.000    3.000    9.000   27.000
 0.016    0.062    0.250    1.000    4.000   16.000   64.000
 0.008    0.040    0.200    1.000    5.000   25.000  125.000
```

Two important results emerge from these patterns:

| | Examples |
|---|---|
| 1. Any number *A* raised to the power 1 is equal to *A*. | `1 2 3 4 5 6*1`<br>`1 2 3 4 5 6` |
| 2. Any number raised to the power 0 is equal to 1. | `1 2 3 4 5 6*0`<br>`1 1 1 1 1 1` ⊟5-6 |

The case of a zero left argument has not been considered. From the foregoing we may conclude that 0*0 should be 1 and that 0*1 should be 0. Further entries in the expression 0*0 1 2 3 4 will be obtained by multiplying by the factor 0 and are all zero:

```
      0*0 1 2 3 4 5
1  0  0  0  0  0
```

Recalling that *A**¯1 was obtained from *A**0 by dividing by *A*, we may now attempt to define a result for 0*¯1 by dividing the value for 0*0 (that is, 1) by the appropriate factor. But this factor is 0, and division by 0 is not allowed. Hence the function 0**R* is not defined for negative values of the right argument *R*.

The application of the power function to a negative <u>left</u> argument is straightforward. Recall that `3*4` is equivalent to `×/4⍴3`, and that in general `A*4` is equivalent to `×/4⍴A`. Hence if `A` is `¯3` we have:

```
      4⍴¯3
¯3  ¯3  ¯3  ¯3
      ×/4⍴¯3
81
     ¯3*4
81
      5⍴¯3
¯3  ¯3  ¯3  ¯3  ¯3
      ×/5⍴¯3
¯243
      ¯3*5
¯243
```

The foregoing results can now be used to construct a table of the power function for both positive and negative arguments, including `0` in the right argument only:

```
      I←(0-⌽⍳4),⍳4
      J←(⍳7)-4
      I
¯4  ¯3  ¯2  ¯1  1  2  3  4
      J
¯3  ¯2  ¯1  0  1  2  3
      I∘.*J
  ¯0.016   0.062  ¯0.250   1.000  ¯4.000  16.000 ¯64.000
  ¯0.037   0.111  ¯0.333   1.000  ¯3.000   9.000 ¯27.000
  ¯0.125   0.250  ¯0.500   1.000  ¯2.000   4.000  ¯8.000
  ¯1.000   1.000  ¯1.000   1.000  ¯1.000   1.000  ¯1.000
   1.000   1.000   1.000   1.000   1.000   1.000   1.000
   0.125   0.250   0.500   1.00C   2.000   4.000   8.000
   0.037   0.111   0.333   1.000   3.000   9.000  27.000
   0.016   0.062   0.250   1.000   4.000  16.000  64.000
```

It should also be recalled that `0*A` is defined for non-negative values of `A`:

```
      0*0 1 2 3 4 5
1  0  0  0  0  0
```

7⊟

## 6.6 THE POWER FUNCTION FOR RATIONAL ARGUMENTS

When the power function is applied to a right argument consisting of successive integers, the successive elements of the result increase by a fixed factor. For example:

```
      4*0 1 2 3 4 5 6 7 8 9
1  4  16  64  256  1024  4096  16384  65536  262144
```

The multiplying factor is 4. This same pattern is observed when the elements of the right argument are equally spaced, even though the spacing is not equal to 1. For example:

```
      4*0 2 4 6 8
1  16  256  4096  65536
```

The multiplying factor is now 16.

The first pattern above can be thought of as being obtained from the second by squeezing the odd integers between the even integers. Hence if the multiplying factor for the pattern 4*0 1 2 3 4 5 6 7 8 9 is 4, the factor for the pattern 4*0 2 4 6 8 must be 4×4, which agrees with the earlier observation.

Similarly the pattern 4*0 .5 1 1.5 2 2.5 3 3.5 4 4.5 5 can be thought of as being obtained by squeezing the entries .5, 1.5, 2.5, 3.5, and 4.5 between the integers 1, 2, 3, 4, and 5. In this case the multiplying factor must be 2, since the product of two factors (that is, 2×2) must be equal to the factor 4 which obtains for the pattern for the integers. Therefore:

```
      4*0 .5 1 1.5 2 2.5 3 3.5 4 4.5 5
1  2  4  8  16  32  64  128  256  512  1024
```

Similarly:

```
      9*0 1 2 3 4 5
1  9  81  729  6561  59049
      9*0 .5 1 1.5 2 2.5 3 3.5 4 4.5 5
1  3  9  27  81  243  729  2187  6561  19683  59049
      25*0 1 2 3 4 5
1  25  625  15625  390625  9765625
      25*0 .5 1 1.5 2 2.5 3 3.5 4 4.5
1  5  25  125  625  3125  15625  78125  390625  1953125      ⌹8
```

Each of the left arguments used above is a perfect square, that is, a number which is equal to some integer multiplied by itself. Thus 4 equals 2×2 and 9 equals 3×3 and 25 equals 5×5. Because of this property, the multiplying factor in each of the "squeezed" patterns is an integer. Since 3 is not a perfect square, a left argument of 3 gives a pattern in which the fractional powers are not integers:

```
      3*0 .5 1 1.5 2 2.5 3
   1.000   1.732   3.000   5.196   9.000  15.588  27.000
```

Nevertheless, the pattern is maintained, the multiplying factor is 1.732 (correct to 3 places) and 1.732×1.732 is (approximately) equal to 3.

From this it appears that `3*.5` is a number which multiplied by itself gives `3`; it is called the <u>square root</u> of `3`. Similarly, `2*.5` is the square root of `2`, and `(2*.5)×(2*.5)` must equal `2`.

The square root of a number can be obtained by "guessing and testing" much like the method described for division at the beginning of Chapter 5. For example, to obtain the square root of `2` we might try `1` (which is too small because `1×1` is less then `2`), and `2` (which is too large since `2×2` is greater than `2`), and then `1.5`. Since `1.5×1.5` is `2.25`, this is also too large. The next trial might be `1.4` (which is slightly too small), and the next might be `1.42`. Better methods are developed in later chapters.

We can now produce a table of powers using right arguments of the form `(⍳N)÷2`:

```
      I←1 2 3 4 5 6 7 8 9
      J←0 .5 1 1.5 2 2.5
      I∘.*J
  1.000    1.000    1.000    1.000    1.000    1.000
  1.000    1.414    2.000    2.828    4.000    5.657
  1.000    1.732    3.000    5.196    9.000   15.588
  1.000    2.000    4.000    8.000   16.000   32.000
  1.000    2.236    5.000   11.180   25.000   55.902
  1.000    2.449    6.000   14.697   36.000   88.182
  1.000    2.646    7.000   18.520   49.000  129.642
  1.000    2.828    8.000   22.627   64.000  181.019
  1.000    3.000    9.000   27.000   81.000  243.000
```

The same reasoning can be applied to right arguments of the form `(⍳N)÷K` for any value of `K`:

```
      (⍳6)÷3
0.333  0.667  1  1.333  1.667  2

      I∘.*(⍳6)÷3
  1.000    1.000    1.000    1.000    1.000    1.000
  1.260    1.587    2.000    2.520    3.175    4.000
  1.442    2.080    3.000    4.327    6.240    9.000
  1.587    2.520    4.000    6.350   10.079   16.000
  1.710    2.924    5.000    8.550   14.620   25.000
  1.817    3.302    6.000   10.903   19.812   36.000
  1.913    3.659    7.000   13.391   25.615   49.000
  2.000    4.000    8.000   16.000   32.000   64.000
  2.080    4.327    9.000   18.721   38.941   81.000
```

```
      (⍳6)÷4
0.25  0.5  0.75  1  1.25  1.5
      I∘.*(⍳6)÷4
   1.000   1.000   1.000   1.000   1.000   1.000
   1.189   1.414   1.682   2.000   2.378   2.828
   1.316   1.732   2.280   3.000   3.948   5.196
   1.414   2.000   2.828   4.000   5.657   8.000
   1.495   2.236   3.344   5.000   7.477  11.180
   1.565   2.449   3.834   6.000   9.391  14.697
   1.627   2.646   4.304   7.000  11.386  18.520
   1.682   2.828   4.757   8.000  13.454  22.627
   1.732   3.000   5.196   9.000  15.588  27.000

      (⍳6)÷5
0.2  0.4  0.6  0.8  1  1.2
      I∘.*(⍳6)÷5
   1.000   1.000   1.000   1.000   1.000   1.000
   1.149   1.320   1.516   1.741   2.000   2.297
   1.246   1.552   1.933   2.408   3.000   3.737
   1.320   1.741   2.297   3.031   4.000   5.278
   1.380   1.904   2.627   3.624   5.000   6.899
   1.431   2.048   2.930   4.193   6.000   8.586
   1.476   2.178   3.214   4.743   7.000  10.330
   1.516   2.297   3.482   5.278   8.000  12.126
   1.552   2.408   3.737   5.800   9.000  13.967
```

The foregoing results have all involved applying the power function to non-integer right arguments and non-negative left arguments. In general it is not possible to apply it to non-integer right arguments together with *negative* left arguments. For example, to evaluate `¯4*.5` it would be necessary to determine a result *R* such that `R×R` equals `¯4`. It is, however, impossible to find such a number, since the product of any number with itself is non-negative. ⌹9-10

# 7

# The Residue Function and Factoring

## 7.1 THE RESIDUE FUNCTION

Consider the following expressions:

```
      3×0 1 2 3 4 5 6
0 3 6 9 12 15 18
      1+3×0 1 2 3 4 5 6
1 4 7 10 13 16 19
      2+3×0 1 2 3 4 5 6
2 5 8 11 14 17 20
```

From the first expression, it is clear that the numbers 0, 3, 6, 9, 12, 15 and 18 are each the product of 3 and some integer; they are therefore said to be integer multiples (or simply multiples) of 3. A number which is an integer multiple of 3 is also said to be divisible by 3.

The numbers 1, 4, 7, 10, 13, 16, and 19 are not divisible by 3; when divided by 3 they each yield an integer quotient and a remainder of 1. Similarly the numbers 2, 5, 8, 11, 14, 17, and 20 each yield a remainder of 2 when divided by 3. The remainder when dividing an integer by 3 must be either 2 or 1 or 0. If the remainder is 0 the number is, of course, divisible by 3.

The remainder obtained on dividing an integer *B* by an integer *A* is a function of *A* and *B*. This function is called the remainder or residue and is denoted by a vertical line as follows: *A*|*B*. For example:

```
      3|6
0
      3|7
1
      3|0 1 2 3 4 5 6 7 8 9 10
0 1 2 0 1 2 0 1 2 0 1
      5|0 1 2 3 4 5 6 7 8 9 10
0 1 2 3 4 0 1 2 3 4 0
```

A function table for residue is shown in Figure 7.1. From this table it should be clear that the results of the expression *A*|*B* must be one of the integers 0, 1, 2, 3, etc., up to *A*-1. That is, the results belong to the vector ¯1+⍳*A*.

| \| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| 4 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 |
| 5 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Left Domain: ⍳8
Right Domain: ⍳14
Body: (⍳8)∘.|⍳14
Symbol: |

Table of Residues

Figure 7.1 ⌹1-2

## 7.2 NEGATIVE RIGHT ARGUMENTS

The following examples show how the residue function applies to negative right arguments:

```
      S←¯6+⍳11
      S
¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
      3×S
¯15 ¯12 ¯9 ¯6 ¯3 0 3 6 9 12 15
      3|3×S
0 0 0 0 0 0 0 0 0 0 0
      1+3×S
¯14 ¯11 ¯8 ¯5 ¯2 1 4 7 10 13 16
      3|1+3×S
1 1 1 1 1 1 1 1 1 1 1
      2+3×S
¯13 ¯10 ¯7 ¯4 ¯1 2 5 8 11 14 17
      3|2+3×S
2 2 2 2 2 2 2 2 2 2 2
```

It should be clear from these examples that the 3-residue of *B* (that is, 3|*B*) is obtained by adding or subtracting some integer multiple of 3 so that the result is the smallest non-negative number that can be so obtained. In general, the result *A*|*B* is the smallest non-negative integer that can be obtained by adding to, or subtracting from, *B* some integer multiple of *A*. ⌹3-4

## 7.3 DIVISIBILITY

The integer *B* is divisible by the integer *A* only if the *A*-residue of *B* is zero, that is, only if (*A*|*B*)=0. Since the expression (⍳8)∘.|0,⍳14 produced a table of residues

(Table 7.1), the expression `0=(⍳8)∘.|0,⍳14` will produce the body of the corresponding divisibility table:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

It is also interesting to arrange the integers 0 to 99 in a 10 by 10 table and then observe the patterns produced by first taking residues and then determining divisibility. For example:

```
      M←(10×0,⍳9)∘.+0,⍳9
      M
 0   1   2   3   4   5   6   7   8   9
10  11  12  13  14  15  16  17  18  19
20  21  22  23  24  25  26  27  28  29
30  31  32  33  34  35  36  37  38  39
40  41  42  43  44  45  46  47  48  49
50  51  52  53  54  55  56  57  58  59
60  61  62  63  64  65  66  67  68  69
70  71  72  73  74  75  76  77  78  79
80  81  82  83  84  85  86  87  88  89
90  91  92  93  94  95  96  97  98  99
```

```
      5|M
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
0 1 2 3 4 0 1 2 3 4
```

```
      0=5|M
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
1 0 0 0 0 1 0 0 0 0
```

```
      3|M
0 1 2 0 1 2 0 1 2 0
1 2 0 1 2 0 1 2 0 1
2 0 1 2 0 1 2 0 1 2
0 1 2 0 1 2 0 1 2 0
1 2 0 1 2 0 1 2 0 1
2 0 1 2 0 1 2 0 1 2
0 1 2 0 1 2 0 1 2 0
1 2 0 1 2 0 1 2 0 1
2 0 1 2 0 1 2 0 1 2
0 1 2 0 1 2 0 1 2 0
```

```
      0=3|M
1 0 0 1 0 0 1 0 0 1
0 0 1 0 0 1 0 0 1 0
0 1 0 0 1 0 0 1 0 0
1 0 0 1 0 0 1 0 0 1
0 0 1 0 0 1 0 0 1 0
0 1 0 0 1 0 0 1 0 0
1 0 0 1 0 0 1 0 0 1
0 0 1 0 0 1 0 0 1 0
0 1 0 0 1 0 0 1 0 0
1 0 0 1 0 0 1 0 0 1
```

## 7.4 FACTORS

If *B* is divisible by *A*, then *A* is said to be a factor of *B*. For example, 3 is a factor of 12, and 5 is a factor of 15, and so on as shown below:

```
      12÷3                       3|12
4                          0

      15÷5                       5|15
3                          0

      9÷3                        3|9
3                          0

      24÷4                       4|24
6                          0

      24÷8                       8|24
3                          0
```

From these examples it is clear that the factors of any number *B* occur in pairs such that the product of the pair is equal to *B*. Thus, if 3 is a factor of 12 then 12÷3 (that is, 4) is also a factor and 3×4 is equal to 12. In general, if *A* is a factor of *B*, then *B*÷*A* is also a factor and the product of the pair of factors *A* and *B*÷*A* (that is, (*B*÷*A*)×*A*) is equal to *B*.

All possible factors of a number *B* can be found by simply trying to divide it by each of the integers from 1 up to and including *B*. For example, the number 24 has the following 8 factors:

```
      1 2 3 4 6 8 12 24
```

The factor pairs of 24 can be obtained by simply dividing 24 by the vector of its factors as follows:

```
      24÷1 2 3 4 6 8 12 24
24 12 8 6 4 3 2 1
```

Thus 1 and 24 are a pair; 2 and 12 are a pair, and so on.

The residue function can be used to determine which of the integers ⍳*B* are factors of *B*. For example, if *B* is 6, then:

```
      1 2 3 4 5 6|6
0 0 0 2 1 0
      0=1 2 3 4 5 6|6
1 1 1 0 0 1
```

The positions of the 1's in the last vector indicate which of the integers 1 2 3 4 5 6 are factors of 6. For example,

since the third element is 1, then 3 is a factor, and since the fourth element is 0, then 4 is not a factor. The vector 1 1 1 0 0 1 can be used to pick out the actual factors 1 2 3 6 by means of the compression function discussed in
13-16 the following section.

## 7.5 COMPRESSION

The following examples show the behavior of the compression function:

```
      1 0 1 0 1/1 2 3 4 5
1 3 5
      1 0 1 0 1/2 3 5 7 11
2 5 11
      (⍳6)|6
0 0 0 2 1 0
      0=(⍳6)|6
1 1 1 0 0 1
      (0=(⍳6)|6)/⍳6
1 2 3 6
      (0=(⍳24)|24)/⍳24
1 2 3 4 6 8 12 24
```

The left argument of compression must be a vector of 1's and 0's and forms a "sieve" which picks up the element of the
17-18 right argument wherever a 1 occurs in the left argument.

## 7.6 PRIME NUMBERS

The following expressions yield all factors for each of the integers from 1 to 8:

```
      (0=(⍳1)|1)/⍳1
1

      (0=(⍳2)|2)/⍳2
1 2

      (0=(⍳3)|3)/⍳3
1 3

      (0=(⍳4)|4)/⍳4
1 2 4

      (0=(⍳5)|5)/⍳5
1 5

      (0=(⍳6)|6)/⍳6
1 2 3

      (0=(⍳7)|7)/⍳7
1 7

      (0=(⍳8)|8)/⍳8
1 2 4 8
```

Any number which has exactly two distinct factors is called a prime number. From the above examples it is clear that 2, 3, 5, and 7 are primes, but 1, 4, 6, and 8 are not. Thus a prime has no factors other than itself and 1.

If *K* is a vector of 0's and 1's, then +/*K* gives a count of the number of 1's in *K*. For example:

```
      +/1 1 0 1 0 0 0 1
4
      0=(ι8)|8
1 1 0 1 0 0 0 1
      +/0=(ι8)|8
4
```

The conditions for a prime number stated above in words can therefore be stated algebraically as follows: *B* is a prime number if the expression 2=+/0=(ι*B*)|*B* has the value 1. For example:

```
      2=+/0=(ι1)|1
0
      2=+/0=(ι2)|2
1
      2=+/0=(ι3)|3
1
      2=+/0=(ι4)|4
0
      2=+/0=(ι5)|5
1
      2=+/0=(ι6)|6
0
      2=0+/=(ι7)|7
1
      2=+/0=(ι8)|8
0
```

This same test can be used to obtain all of the primes up to a certain value by applying it to a divisibility table. Consider, for example, the following tables:

| \| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 |
| 4 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 |
| 5 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 |
| 6 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 | 3 | 4 | 5 | 0 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 |
| 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 0 | 1 | 2 | 3 |
| 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 11 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 0 | 1 |
| 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 |

Left D:ι12
Right D:ι12
Body:(ι12)∘.|ι12
Symbol:|

| *D* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Left D:ι12
Right D:ι12
Body:0=(ι12)∘.|ι12
Symbol:*D*

The last table shows divisibility. For example, the 1's in the 6th column show the position of the 4 factors of 6. Therefore the sum of the 6th column tells how many factors 6 has, and similarily for each column. The sum of the columns is obtained by summing the rows of the transpose of the table. Thus:

```
      +/⍉0=(⍳12)∘.|⍳12
1 2 2 3 2 4 2 4 3 4 2 6
```

The last result above gives the number of factors for each of the numbers 1 to 12. Therefore the expression 2=+/⍉0=(⍳12)∘.|⍳12 determines which numbers are primes:

```
      2=+/⍉0=(⍳12)∘.|⍳12
0 1 1 0 1 0 1 0 0 0 1 0
```

This vector of 0's and 1's can be used to compress the vector ⍳12 to finally pick out all of the primes up to 12:

```
      (2=+/⍉0=(⍳12)∘.|⍳12)/⍳12
```

19-24⍈ 
```
2 3 5 7 11
```

# 8 Monadic Functions

## 8.1 INTRODUCTION

Each of the functions discussed thus far have applied to two quantities. Thus in the expressions 3×4 and 3+4 and 3⌈4, each of the functions ×, +, and ⌈ apply to the two quantities 3 and 4. Recall that these quantities are called the *arguments* of the function; the one to the left of the function is called the *first* or *left* argument, and the one to the right is called the *second* or *right* argument.

A function having two arguments is said to be *dyadic*, the prefix *dy* meaning two. There are also functions which apply to one argument; they are called *monadic* functions. The following examples show a monadic function which is called the *factorial* function:

```
      !1
1

      !2
2

      !3
6

      !4
24

      !5
120

      !6
720

      !7
5040

      !8
40320
```

From the examples it should be clear that factorial 3 is the product of the factors 1 2 3, factorial 4 is the product of the factors 1 2 3 4, and so on. The examples also illustrate a point which applies to all monadic functions: the symbol for the function (in this case, !) precedes its single argument.

The argument of a monadic function may (like the arguments of a dyadic function) be a vector. For example:

```
      !1 2 3 4 5 6 7 8
1 2 6 24 120 720 5040 40320
```

⍠1-2

## 8.2 NEGATION

Negation is a monadic function denoted by the symbol -. For example:

```
      -3
¯3
      -5
¯5
      -¯5
5
      --5
5
      -2 3 5.8
¯2 ¯3 ¯5.8
```

```
      X←3
      -X
¯3
      S←2 3 5
      -S
¯2 ¯3 ¯5
      --S
2 3 5
```

From these examples it should be clear that negation of a number *B* is equivalent to subtracting *B* from zero; that is, -*B* is equivalent to 0-*B*. In other words, negation changes the sign of its argument.

It is also apparent from the examples that the symbol used for the monadic function of negation is the same as that already used for the dyadic function of subtraction. This might be expected to cause confusion, but it does not. For example:

```
      4-3
1
      4×-3
¯12
      4--3
7
```

Thus the symbol - denotes subtraction if it is preceded by an argument, but denotes negation if it is preceded by a function.

This double use of symbols (once for a dyadic function and once for a monadic function) will be applied to many other symbols as well as the -. For example, +, ×, ÷, ⌈, ⌊, and |, already used for dyadic functions, will be used to
3⊟ denote monadic functions as well.

## 8.3 RECIPROCAL

The <u>reciprocal</u> function is a monadic function denoted by ÷ and defined as follows: ÷*B* is equal to 1÷*B*. For example:

```
      ÷2
0.5
      ÷4
0.25
```

```
      S←⍳10
      S
1 2 3 4 5 6 7 8 9 10
      R←÷S
      R
1 0.5 0.33333 0.25 0.2 0.16667 0.14286 0.125 0.11111 0.1
      S×R
1 1 1 1 1 1 1 1 1 1
```

⌹4-5

## 8.4 MAGNITUDE

The numbers 5 and ¯5 are said to have the same size or *magnitude*, namely 5. In other words, the magnitude of a number is a function (denoted by |) which ignores the sign of the number. For example:

```
      |5
5
      |¯5
5
      S←¯6+⍳11
      S
¯5 ¯4 ¯3 ¯2 ¯1 0 1 2 3 4 5
      |S
5 4 3 2 1 0 1 2 3 4 5
      T←6 ¯3 2 ¯5 4
      T÷T
1 1 1 1 1
      T÷|T
1 ¯1 1 ¯1 1
```

⌹6

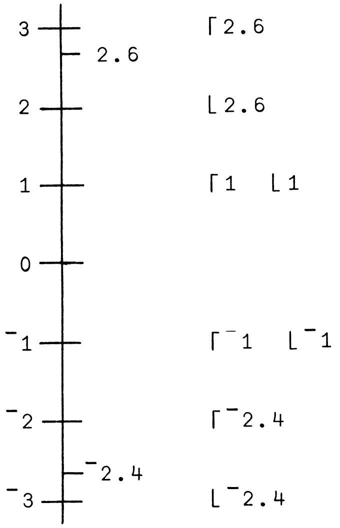
## 8.5 FLOOR AND CEILING

The *floor* function is denoted by ⌊ and yields the next integer just below or equal to the argument. The *ceiling* function is denoted by ⌈ and yields the next integer just above or equal to the argument. For example:

```
      ⌊3
3

      ⌊3.14
3

      ⌊¯3.14
¯4

      ⌊¯3
¯3

      ⌊¯1.5 ¯1 ¯.5 0 .5 1 1.5
¯2 ¯1 ¯1 0 0 1 1
```

```
      ⌈3
3

      ⌈3.14
4

      ⌈¯3.14
¯3

      ⌈¯3
¯3

      ⌈¯1.5 ¯1 0 .5 1 1.5
¯1 ¯1 0 1 1 2
```

The floor and ceiling functions are easily visualized by drawing the integers as the floors (and ceilings) in a building as follows:




The following examples illustrate how the monadic function floor is related to the dyadic function residue:

```
      17÷5
3.4
      ⌊17÷5
3
      (17-5|17)÷5
3
```

## 8.6 COMPLEMENT

The complement function is denoted by ~ and applies only to logical arguments (that is, 0 and 1). When applied to 0 it produces 1, and when applied to 1 it produces 0. For example:

```
      ~1
0
      ~0
1
      ~1 0 1 0 1 1
0 1 0 1 0 0
      0=3|⍳12
0 0 1 0 0 1 0 0 1 0 0 1
      ~0=3|⍳12
1 1 0 1 1 0 1 1 0 1 1 0
      (~0=3|⍳12)/⍳12
1 2 4 5 7 8 10 11
```

```
      (0≠3|⍳12)/⍳12
1 2 4 5 7 8 10 11
```

The symbol ~ is called <u>tilde</u>. ⌹8-10

## 8.7 RAVEL

<u>Ravel</u> is a monadic function (denoted by a comma) which ravels a table to produce a vector which contains the elements of the table in order by rows. For example:

```
      T←2 3 5∘.×⍳6
      T
2   4   6   8  10  12
3   6   9  12  15  18
5  10  15  20  25  30
      ,T
2 4 6 8 10 12 3 6 9 12 15 18 5 10 15 20 25 30
```

The ravel function applied to any vector simply yields the vector unchanged. The result of ravel is always a vector, therefore when applied to a scalar it produces a one-element vector whose element is equal to the scalar. Although a scalar and a one-element vector are very similar, they possess certain essential differences. In particular, the vector can be indexed but the scalar cannot. For example:

```
      S←3
      V←,3
      V[1]
3
      S[1]
RANK ERROR
      S[1]
      ∧
```

⌹11

## 8.8 SIZE

The number of elements in a vector *V* is called the <u>size</u> of the vector. Size is therefore a monadic function and is denoted by ⍴. For example:

```
      V←2 3 5 7 11
      ⍴V
5
      X←⍳7
      ⍴X
7
      ⍴X[2 3 5]
3
      ⍴X[⍳2]
2
```

When applied to a table, the function ρ yields a two-element vector giving the number of rows in the table followed by the number of columns. For example:

```
      T←2 3 5∘.×⍳7
      ρT
3 7
      ρ⍉T
7 3
```

The result of the expression ρ*A* is a vector with as many elements as there are indices to *A*. For example, the table *T* takes two indices (as in the expression *T*[3;4]) and ρ*T* has two elements as shown in the preceding paragraph. Since a scalar takes <u>no</u> indices, the result of applying the size function ρ to a scalar is an <u>empty</u> vector having no elements.

The expression ρ,*A* yields a one-element vector whose element is equal to the total number of elements in *A*, regardless of whether *A* is a scalar, a vector, or a table. For example:

```
      ρT
3 7
      ρ,T
21
      ρV
5
      ρ,V
5
      S←2
      ρS

      ρ,S
12 1
```

# 9 Function Definition

## 9.1 INTRODUCTION

The expression `0=3|X` was shown (in Chapter 7) to determine whether the argument *X* was divisible by 3. For example:

```
      0=3|9
1
      0=3|10
0
```

The expression `0=3|X` is therefore a monadic function of *X* in the sense that for any particular value assigned to *X*, the expression yields a particular corresponding value.

Unlike the functions floor, ceiling, and magnitude (which have the symbols ⌊, ⌈, and |), the function determined by the expression `0=3|X` has no special single symbol to denote it. It would, of course, be impractical to assign a special symbol to every possible such expression. However, it is important to be able to assign a name to any such expression which happens to be of interest at the moment, and then be able to use that name for the function just as ⌊, ⌈, and | are used for the floor, ceiling, and magnitude functions.

The name *DT* is assigned to the function determined by the expression `0=3|X` in the following manner:

```
      ∇Z←DT X
      Z←0=3|X ∇
```

The above is called definition of the function *DT*. Once the function *DT* has been so defined, it can be used like any other monadic function as follows:

```
      DT 9
1
      DT 10
0
      DT ⍳10
0 0 1 0 0 1 0 0 1 0
```

The symbol ∇ which begins and ends a function definition is called del.

Any number of such functions may be defined, but they must, of course, be given distinct names. These function names, like the names introduced for values in Chapter 1, must begin with a letter but may include both letters and digits. For example:

```
      ∇Z←D4 X
      Z←0=4|X ∇
      D4 ι10
0 0 0 1 0 0 0 1 0 0
      ∇Z←D5 X
      Z←0=5|X ∇
      D5 ι10
0 0 0 0 1 0 0 0 0 1
      ∇Z←Q X
      Z←(X-3)×(X-5) ∇

      Q 6
3
      Q 7
8
      Q ι7
8 3 0 ¯1 0 3 8
```

The rules for determining the meaning of a function definition are very simple: when the function is applied to an argument, that argument is substituted for each occurrence of the name *X* in the second line of the function definition, and the result thereby assigned to the name *Z* is the result of the function. For example, to evaluate *Q* 7, the 7 is substituted for *X* to yield

```
      Z←(7-3)×(7-5)
```

This is evaluated to yield the result 8. Hence:

```
      Q 7
8
```

1-4

Functions such as floor and ceiling which have been assigned special fixed symbols will now be called primitive functions in order to distinguish them from the new class of defined functions just introduced. A defined function can be used within expressions, just as primitives are. For example:

```
      Q 6
3
      4×Q 6
12
      DT 12
1
```

```
      DT 4×Q 6
1
      Q Q 6
0
```

⌹5-7

## 9.2 DEFINITION OF DYADIC FUNCTIONS

The expression `0=X|Y` determines whether the argument *X* is a factor of the argument *Y*. For example:

```
      0=5|9
0
      0=7|21
1
```

The expression `0=X|Y` is therefore a dyadic function of the arguments *X* and *Y* in the sense that for any particular values of *X* and *Y* the expression yields a particular corresponding value.

The name *F* is assigned to the dyadic function determined by the expression `0=X|Y` in the following manner:

```
      ∇Z←X F Y
      Z←0=X|Y ∇
```

The function *F* can now be applied to pairs of arguments as illustrated below:

```
      5 F 9
0
      7 F 21
1
      5+7 F 21
6
      (5×7) F (5×21)
1
```

⌹8-13

## 9.3 A FUNCTION TO GENERATE PRIMES

In Chapter 7 it was shown that the expression

```
      (2=+/⍉0=(⍳N)∘.|⍳N)/⍳N
```

would produce a vector of all the primes up to the integer *N*. Therefore a function *PR* can be defined to generate primes as follows:

```
      ∇Z←PR X
      Z←(2=+/⍉0=(⍳X)∘.|⍳X)/⍳X∇
```

The following examples show the use of the function *PR*:

```
      PR 12
2 3 5 7 11
      +/PR 12
28
      PR 55
2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53
```

## 9.4 TEMPERATURE SCALE CONVERSION FUNCTION

The Centigrade scale and the Fahrenheit scale are two different scales for measuring temperature. For any given temperature reading in Centigrade there is therefore a corresponding value in Fahrenheit; in other words, the Fahrenheit value is a function of the Centigrade value. This function will be expressed as a defined function called *CTOF* (for Centigrade TO Fahrenheit).

The Centigrade scale has 100 degrees between the freezing and boiling points of water, whereas the Fahrenheit scale has 180 degrees between these same points. Therefore any Centigrade reading *X* must be multiplied by 180 and divided by 100: that is, 180×*X*÷100. Moreover, 0 degrees Centigrade (the freezing point of water) corresponds to 32 degrees Fahrenheit and so it is necessary to add 32 to the foregoing expression, giving 32+180×*X*÷100. The conversion function *CTOF* may therefore be defined and used as follows:

```
      ∇Z←CTOF X
      Z←32+180×X÷100 ∇

      CTOF 0
32
      CTOF 100
212
      CTOF ¯40 ¯20 0 20 40 60 80 100
¯40 ¯4 32 68 104 140 176 212
```

The function *CTOF* determines the Fahrenheit value as a function of the Centigrade value. It is, of course, also possible to define a function *FTOC* which determines the Centigrade value as a function of the Fahrenheit value:

```
      ∇Z←FTOC X
      Z←100×(X-32)÷180 ∇

      FTOC ¯40 ¯4 32 68 104 140 176 212
¯40 ¯20 0 20 40 60 80 100
      CTOF FTOC ¯40 ¯4 32 68 104 140 176 212
¯40 ¯4 32 68 104 140 176 212
      FTOC CTOF ¯40 ¯20 0 20 40 60 80 100
¯40 ¯20 0 20 40 60 80 100
```

The last two lines above illustrate the fact that the function *FTOC* undoes the work of *CTOF*, and the preceding two lines illustrate that *CTOF* undoes the work of *FTOC*. The functions *FTOC* and *CTOF* are therefore inverse functions. ⌹15

## 9.5 FUNCTIONS ON RATIONALS

If *X* is a vector of two integer elements and *Y* is a vector of two integer elements, then ÷/*X* is a rational and ÷/*Y* is a rational. Moreover, as shown in Section 5.7, the product (÷/*X*)×(÷/*Y*) is equal to ÷/(*X*×*Y*). Therefore, the following function multiplies two rationals to produce the two element vector which represents their product:

```
      ∇Z←X P Y
      Z←X×Y ∇
```

For example:

```
      3 4 P 7 5
21 20
      ÷/3 4 P 7 5
1.05
      (÷/3 4)×(÷/7 5)
1.05
```

Similarly, the following function will add rationals:

```
      ∇ Z←X A Y
      Z←(+/X×⌽Y),X[2]×Y[2] ∇
```

For example:

```
      3 4 A 7 5
43 20
      ÷/3 4 A 7 5
2.15
      (÷/3 4)+(÷/7 5)
2.15
```

⌹16-18

## 9.6 TRACING FUNCTION EXECUTION

A function can be defined by a single expression (as in the examples thus far), or it can be defined by a sequence of expressions. For example:

```
    ∇ Z←R X
[1]   T1←4×X
[2]   T2←3×X*2
[3]   T3←2×X*3
[4]   Z←T1+T2+T3∇
```

```
      R 2
36
      R 2 3 4
36 93 192
```

The statements are executed in the order in which they appear on the page, and each is identified by its number appearing in brackets on the left.

To understand the behavior of a function it is often helpful to examine some of the intermediate results produced by each of the individual statements in its definition. To indicate that each intermediate result produced in executing the function *R* is to be displayed, we would write

```
      TΔR←1 2 3 4
```

Thereafter, the execution of *R* would be accompanied by a display of the intermediate results as follows:

```
      W←R 2
R[1]   8
R[2]   12
R[3]   16
R[4]   36

      W
36
      W←R 2 3 4
R[1]   8 12 16
R[2]   12 27 48
R[3]   16 54 128
R[4]   36 93 192

      W
36 93 192
```

Such a display of the steps of execution of a function is called a trace of the function. The name *TΔR* used in initiating the trace of the function *R* denotes the trace control vector for *R*. In the preceding example, *TΔR* was set to trace every line of *R*, but it could be set to trace only some of them. For example:

```
      TΔR←1 3
      W←R 2 3 4
R[1]   8 12 16
R[3]   16 54 128
```

Moreover, if *TΔR* is set to 0, no tracing is performed:

```
      TΔR←0
      W←R 2 3 4
      W
36 93 192
```

19

# 10

# The Analysis of Functions

## 10.1 INTRODUCTION

The problem of converting temperatures from the Centigrade to the Fahrenheit scale, which was handled by the function *CTOF* of Chapter 9, is often handled by simply providing a table covering the values of interest. For example, Table 10.1 would suffice for a range of temperatures just above the freezing point of water:

| *C* | *F* |
|---|---|
| 0 | 32 |
| 1 | 33.8 |
| 2 | 35.6 |
| 3 | 37.4 |
| 4 | 39.2 |
| 5 | 41 |
| 6 | 42.8 |
| 7 | 44.6 |
| 8 | 46.4 |
| 9 | 48.2 |
| 10 | 50 |

A Table Representation of the Function
*CTOF* for Centigrade Values Near Zero
Table 10.1

It is often more convenient to use such a table than to evaluate the expression $32+180\times C\div 100$ (used in the definition of the function *CTOF*) for each conversion. However, such a tabular representation of a function also has its disadvantages; it provides only a limited set of values and could not, for example, be used directly to find the Fahrenheit equivalent of 25 *C* (which lies outside of the tabled values) or of 5.64 degrees Centigrade (which lies between two of the tabled values). For this reason it is often desirable to determine from such a table the algebraic expression which would produce the same function as that represented by the table.

To appreciate the problem of deriving an algebraic expression for a function represented only by a table, suppose that the expression $32+180\times C\div 100$ is <u>not</u> known and that the only information known about the function is that contained in Table 10.1. One might begin by observing that each Fahrenheit value is at least 32 more than the corresponding Centigrade value, and therefore guess that the desired function is approximately $32+C$. The next step is to append to Table 10.1 a column of values for the function $32+C$ so that they can be compared with the tabled values of $F$:

| $C$ | $F$ | $32+C$ |
|---|---|---|
| 0 | 32 | 32 |
| 1 | 33.8 | 33 |
| 2 | 35.6 | 34 |
| 3 | 37.4 | 35 |
| 4 | 39.2 | 36 |
| 5 | 41 | 37 |
| 6 | 42.8 | 38 |
| 7 | 44.6 | 39 |
| 8 | 46.4 | 40 |
| 9 | 48.2 | 41 |
| 10 | 50 | 42 |

Although the first entries in the columns $F$ and $32+C$ agree (both are 32), the second entry falls short by 0.8, the third entry by 1.6, etc. It therefore appears that one should add $0.8\times C$ to the expression $32+C$, yielding $32+C+.8\times C$ or, more simply, $32+1.8\times C$. If a column of values for $32+1.8\times C$ is appended to the foregoing table and compared with the column $F$ it will be seen that this is the required expression.

The process of determining an expression for a function from a table of the function will be referred to as analyzing the table or, alternatively, as analyzing the function represented by the table. The analysis of tables is not only an interesting puzzle, it is also a problem of the greatest importance, since it underlies every scientific discipline. The reason is that in every area of science and technology, one attempts to determine the functional relationships between various quantities of interest. Thus one wishes to know how the acceleration of an automobile depends on the power of the engine, how the gasoline consumption depends on the speed, how the length of life of the brakes depends on the area of the brake-shoes, how the electric current supplied to the headlamps depends on the battery voltage, how the weight limit of a suspension bridge depends on the size of the cable used, and so on. Moreover, it is important to be able to express these relations algebraically so that it becomes easy to calculate any new values needed.

However, the relationships between two quantities are normally determined by experiments in which the corresponding values of the quantities of interest are measured. Such experiments can only yield a table of values; they do not yield an algebraic expression for the function. The algebraic function must be determined by analysis of the table.

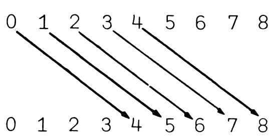
In practice one might do a few experiments, make a small table, derive from it an algebraic expression for the functional relationship, and then do a few more experiments to test (and perhaps revise) the derived expression. In a book this process cannot be simulated completely since we can only give fixed tables resulting from certain experiments, and cannot allow the reader to choose the values to be included in these tables. However, if a computer is available, one person (the teacher) can enter the definition of any function so that another person (the student) can "experiment" with the function at will by simply applying it to any desired arguments. If the student is not permitted to see the original definition of the function, then he can be given the problem of experimenting with the function, determining a table, and deriving from it his own definition of (i.e., algebraic expression for) the function.

The remainder of this chapter will be devoted to the analysis of tables. Three methods are treated: maps, graphs, and difference tables. Difference tables provide the most powerful method of the three, but maps and graphs are treated first because they are easier to comprehend and because maps have already been used for other purposes in earlier chapters. A fourth and more powerful method (called *curve-fitting*) is treated in Chapter 19. ⌹1

## 10.2 MAPS

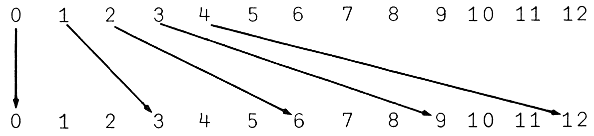
If one first makes a map of a table, then the map can be used as a guide in the analysis of the table. In order to see what guidance the map can provide, it is useful to recall the maps of two simple functions.

If *X*←0,⍳4, then the map of the function 4+*X* against *X* appears as follows:



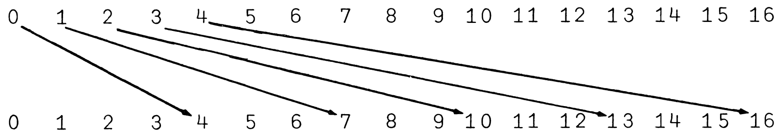

From this it is clear that the addition of a constant (in this case 4) appears in the map as a uniform translation, that is, each point is moved by the same amount, and the mapping arrows all have the same slope.

If, as before, *X*←0,ι4, then the map of the function 3×*X* appears as follows:




From this it is clear that multiplication by a constant (in this case 3) appears in the map as a uniform spreading, that is, the distance between the successive arrowheads (in this case 3) is the constant of multiplication.

Consider now the mapping of a function which involves both addition and multiplication, say 4+3×*X*:




The effects of uniform translation and uniform spreading are now superimposed, but it is still possible to recognize the individual effects of each. These observations will now be applied to the analysis of the function shown in Figure 10.2.

| *X* | *Y* |
|---|---|
| 2 | 1 |
| 3 | 3 |
| 4 | 5 |
| 5 | 7 |
| 6 | 9 |
| 7 | 11 |

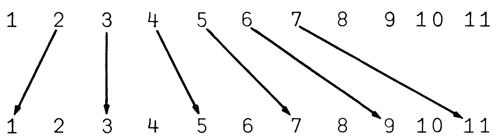
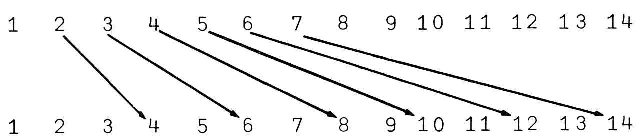


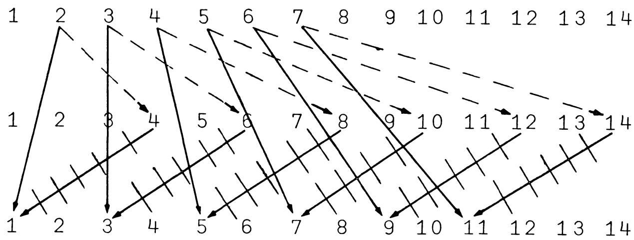

Table and Map of a Function

Figure 10.2

It is usually best to try to account for the multiplication (spreading) first. In this case adjacent arrowheads are separated by 2 units and so the multiplication factor is 2. Therefore we make a map of the function 2×*X* as follows:



The map of 2×*X* is now combined with the map of the original table as follows:




In this map, the original table is represented by normal lines as usual, and the approximating function 2×*X* is represented by broken lines. The scored lines lead <u>from</u> the results of 2×*X* <u>to</u> the results of the tabled function and therefore represent the function that must be applied to the function 2×*X* to yield the tabled function. Since the scored lines all have the same slope, this function must be a translation (by ¯3), that is, the addition of ¯3. The required function is therefore ¯3+2×*X*, as may be verified by computing the values for the case *X*←2 3 4 5 6 7 and comparing them with the second column of Figure 10.2.

The functions analyzed by maps thus far have all been of the form *A*+*B*×*X* where *A* and *B* are constants. In the analysis of more complex functions (such as 3+(5×*X*)+(2×*X*⋆2)), maps are of little help and one of the other methods should be used. ⌹2

## 10.3 GRAPHS

Each row of a function table such as Table 10.1 consists of a pair of numbers representing an argument and a corresponding function value. Any other way of showing the pairing of the numbers in each of the rows is obviously a possible way of representing the function. For example, in a map, each pairing is shown by an arrow from the argument to the corresponding function value.

Any single number can be represented by marking off the integers at equal intervals along a line and then placing a cross on the line to show the desired value. For example 4 might by represented as follows:

```
----------------x--------
|   |   |   |   |   |   |
0   1   2   3   4   5   6
```

A whole set of numbers could be represented by a set of crosses on such a line as follows:

```
------x-x-x-x---x--------
|   |   |   |   |   |   |
0   1   2   3   4   5   6
```

This line represents the set of arguments of the function defined by Table 10.3.

| $X$ | $Y$ |
|---|---|
| 1.5 | 5.5 |
| 2.0 | 4.5 |
| 2.5 | 3.5 |
| 3.0 | 2.5 |
| 4.0 | 0.5 |

Table of a Function

Table 10.3

If the set of function values $Y$ of Table 10.3 are now represented similarly along a vertical line rising from the 0-point of the first line, the picture appears as follows:

```
6 - |
    |
    x
    |
5 - |
    |
    x
    |
4 - |
    |
    x
    |
3 - |
    |
    x
    |
2 - |
    |
    |
    |
1 - |
    |
    x
    |
0 - |-----x-x-x-x---x--------
    |   |   |   |   |   |   |
    0   1   2   3   4   5   6
```

If vertical lines are drawn through the crosses on the horizontal line, and if horizontal lines are drawn through the crosses on the vertical line, the picture appears as follows:

```
6 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---+--------
    |     | | | |   |
5 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---+--------
    |     | | | |   |
4 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---+--------
    |     | | | |   |
3 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---+--------
    |     | | | |   |
2 - |     | | | |   |
    |     | | | |   |
    |     | | | |   |
    |     | | | |   |
1 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---+--------
    |     | | | |   |
0 - +-----x-x-x-x---x--------
    |   |   |   |   |   |   |
    0   1   2   3   4   5   6
```

The pairing of each argument with its particular function value can now be shown by placing a point at the intersection of the lines through them as follows:

```
6 - |     | | | |   |
    |     | | | |   |
    x-----o-+-+-+---+--------
    |     | | | |   |
5 - |     | | | |   |
    |     | | | |   |
    x-----+-o-+-+---+--------
    |     | | | |   |
4 - |     | | | |   |
    |     | | | |   |
    x-----+-+-o-+---+--------
    |     | | | |   |
3 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-o---+--------
    |     | | | |   |
2 - |     | | | |   |
    |     | | | |   |
    |     | | | |   |
    |     | | | |   |
1 - |     | | | |   |
    |     | | | |   |
    x-----+-+-+-+---o--------
    |     | | | |   |
0 - +-----x-x-x-x---x--------
    |   |   |   |   |   |   |
    0   1   2   3   4   5   6
```

In practice, one actually draws neither the lines nor the crosses, but simply marks the points of intersection, producing the following less cluttered picture:

```
6 - |
    |
    |     o
    |
5 - |
    |
    |       o
    |
4 - |
    |
    |         o
    |
3 - |
    |
    |           o
    |
2 - |
    |
    |
    |
1 - |
    |
    |               o
    |
0 - +-------------------------
    |   |   |   |   |   |   |
    0   1   2   3   4   5   6
```

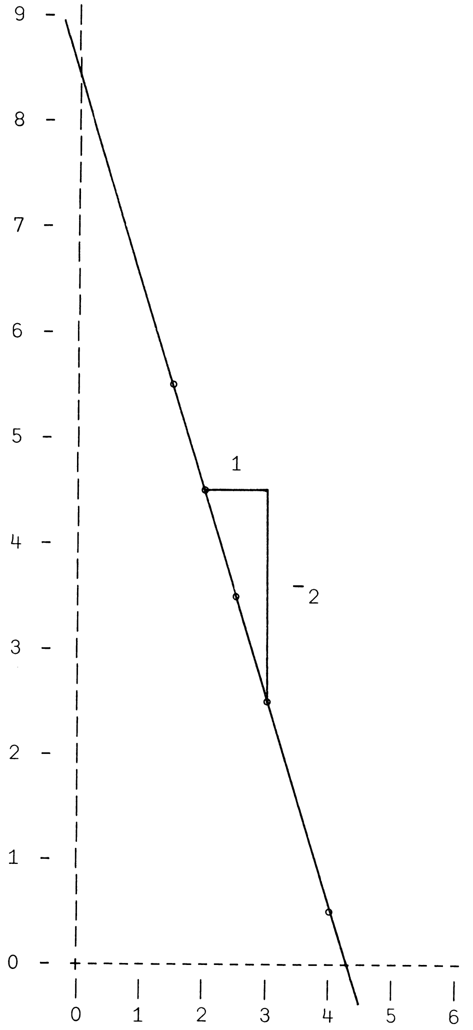
This picture is called a *graph* or *plot* of the function of Table 10.3. Negative values are included by simply extending the horizontal line leftward from the zero and the vertical line downward from the zero.

The vertical line of the graph (which passes through the zero point of the horizontal line) is called the *vertical axis* or *Y-axis*, and the horizontal line (through the zero of the vertical line) is called the *horizontal axis* or *X-axis*. The names are derived from the (arbitrary) convention that the argument of a function is often called $X$ and the result is often called $Y$, so that the expression for a function is in the form $Y \leftarrow F\ X$. ⎕3-4

## 10.4 INTERPRETING A LINEAR GRAPH

If a ruler is laid along the points in the preceding graph, the points will be seen to lie in a straight line. If one graphs a number of functions of the form $A+B\times X$ (where $A$ and $B$ are fixed values), it will be seen that the points in the graph of any such function lie in a straight line. Conversely, every graph whose points all lie in one straight line represents a function of the form $A+B\times X$. Moreover, the values of $A$ and $B$ can be easily determined *from the graph*.

Consider, for example, Figure 10.4 which shows the graph of the function of Table 10.3 with a line drawn through the points. Any point on the line (not only the five taken from the table) represents a point of the function. For example, if the argument $X$ is 1, then the function value $Y$ is 6.5, and if $X$ is 0, then $Y$ is 8.5. But if $X$ is 0, the value of the expression $A+B\times X$ is simply $A$. Hence, for this function $A$ must have the value 8.5.




Graph of Function of Table 10.3

Figure 10.4

Moreover, *B* is clearly the amount that the function changes when the argument is changed from some value to a value greater by 1. Since the function is equal to 4.5 for *X*=2 and is equal to 2.5 for *X*=3 this change is equal to 2.5-4.5 or ¯2. Therefore *B* is equal to ¯2. Finally, the expression for the function must be 8.5+¯2×*X*. This may be verified by evaluating the expression for the values *X*←1.5 2 2.5 3 4 and comparing the results with the second column of Table 10.3.

To summarize, the values of *A* and *B* can be determined from a straight-line graph as follows:

(1) The value of *A* is the height at which the graph line crosses the vertical axis (where *X*=0).

(2) The value of *B* is the change in height corresponding to a change of 1 on the horizontal axis. ⌸5-6

A function table whose graph does not form a straight line is not as easy to interpret as a straight line graph. However, the graph can still provide some guidance.

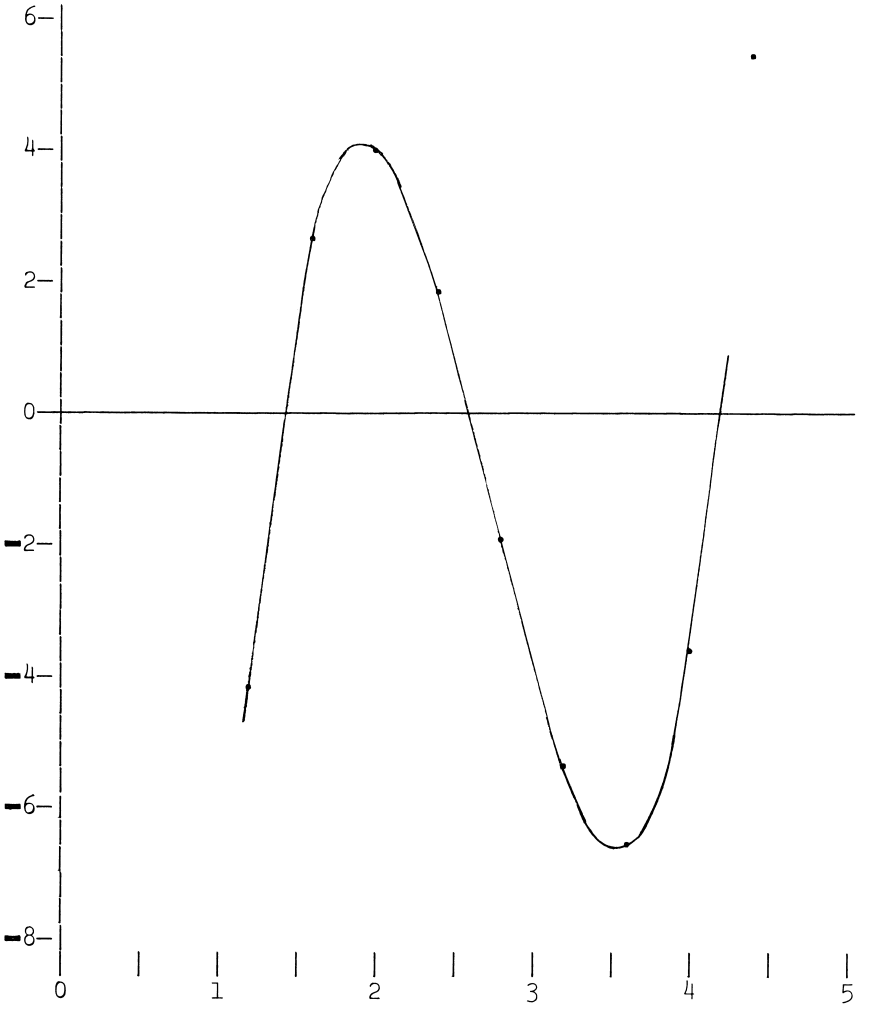
Consider, for example, Figure 10.5 which shows a function table and the corresponding graph. The points do not lie in a straight line, but have been joined by a smooth curve which suggests the function values which should be obtained between the points included in the table itself. For example, the argument 3 is not included in the table, but the curve indicates that the corresponding function value should be approximately ¯3.8.

A number of interesting characteristics of the function can be seen clearly in its graph. For example, it is clear that the function reaches a local low point for an argument value of *X* equal to approximately 3.5 and that it reaches a local high point for a value of *X* a little less than 2. Moreover, it is easy to spot those argument values for which the function has a zero value, namely for *X* equal to (approximately) 1.4 or 2.6 or 4.2.

Since *X*-1.4 is zero for *X*=1.4 and *X*-2.6 is zero for *X*=2.6 and *X*-4.2 is zero for *X*=4.2, then the expression

    (X-1.4)×(X-2.6)×(X-4.2)

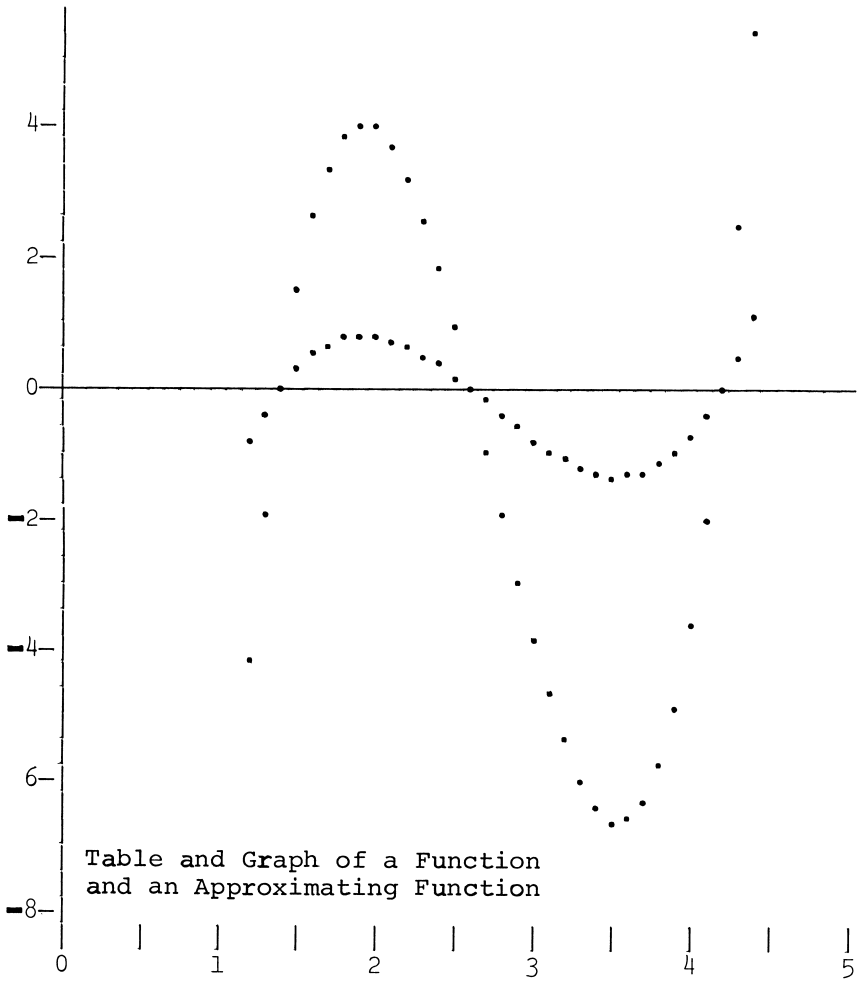
is zero for *X* equal to either 1.4 or 2.6 or 4.2. Hence it will agree with the given function at least for these three values of the argument *X*. In order to see how well this expression agrees with the given function for all points, it can be graphed together with the given function as shown in Figure 10.6.




| $X$ | $Y$ |
|---|---|
| 1.2 | $^-$4.20 |
| 1.6 | 2.60 |
| 2.0 | 3.96 |
| 2.4 | 1.80 |
| 2.8 | $^-$1.96 |
| 3.2 | $^-$5.40 |
| 3.6 | $^-$6.60 |
| 4.0 | $^-$3.64 |
| 4.4 | 5.40 |

Table and Graph of a Function

Figure 10.5




| $X$ | $Y$ | $(X-1.4)$ $\times(X-2.6)$ $\times(X-4.2)$ |
|---|---|---|
| 1.20 | ¯4.20 | ¯0.84 |
| 1.30 | ¯1.88 | ¯0.38 |
| 1.40 | 0.00 | 0.00 |
| 1.50 | 1.48 | 0.30 |
| 1.60 | 2.60 | 0.52 |
| 1.70 | 3.37 | 0.67 |
| 1.80 | 3.84 | 0.77 |
| 1.90 | 4.02 | 0.80 |
| 2.00 | 3.96 | 0.79 |
| 2.10 | 3.67 | 0.73 |
| 2.20 | 3.20 | 0.64 |
| 2.30 | 2.56 | 0.51 |
| 2.40 | 1.80 | 0.36 |
| 2.50 | 0.93 | 0.19 |
| 2.60 | 0.00 | 0.00 |
| 2.70 | ¯0.97 | ¯0.19 |
| 2.80 | ¯1.96 | ¯0.39 |
| 2.90 | ¯2.92 | ¯0.58 |
| 3.00 | ¯3.84 | ¯0.77 |
| 3.10 | ¯4.67 | ¯0.93 |
| 3.20 | ¯5.40 | ¯1.08 |
| 3.30 | ¯5.98 | ¯1.20 |
| 3.40 | ¯6.40 | ¯1.28 |
| 3.50 | ¯6.61 | ¯1.32 |
| 3.60 | ¯6.60 | ¯1.32 |
| 3.70 | ¯6.32 | ¯1.26 |
| 3.80 | ¯5.76 | ¯1.15 |
| 3.90 | ¯4.87 | ¯0.97 |
| 4.00 | ¯3.64 | ¯0.73 |
| 4.10 | ¯2.02 | ¯0.40 |
| 4.20 | 0.00 | 0.00 |
| 4.30 | 2.46 | 0.49 |
| 4.40 | 5.40 | 1.08 |

Figure 10.6

A comparison of the two curves in Figure 10.6 shows that they have the same general shape, that is, the values for the given function appear to be larger than those of the approximating expression by a fixed ratio. A value for this ratio can be determined from two corresponding points, say for an argument value of 2.4. The two corresponding function values are seen to be 1.8 and .36, and the ratio is therefore 1.8÷.36, that is, 5.

A better approximation to the given function is therefore given by 5 times the expression just tried, that is:

```
      5×(X-1.4)×(X-2.6)×(X-4.2)
```

Evaluation of this function for each of the argument values appearing in the first column of Table 10.5 shows that it
7-8⌹ agrees exactly with the function given in the second column.

## 10.5 THE TAKE AND DROP FUNCTIONS

The dyadic functions take and drop are denoted by ↑ and ↓, respectively. The following expressions illustrate their use:

```
      Y←0 1 4 9 16 25 36

      3↑Y              3↓Y
0 1 4            9 16 25 36
      2↑Y              2↓Y
0 1              4 9 16 25 36
      ¯3↑Y             ¯3↓Y
16 25 36         0 1 4 9
      ¯2↑Y             ¯2↓Y
25 36            0 1 4 9 16
```

The take function takes from its right argument the number of elements determined by the left argument, beginning at the front end if the left argument is positive and at the back end if it is negative. The drop function behaves similarly, dropping the indicated number of elements from the right argument.

If the left argument is greater than the number of elements of the right argument, then the extra positions are filled with zeros. For example:

```
      X←2 3 5 7
      6↑X
2 3 5 7 0 0
      ¯6↑X
0 0 2 3 5 7
```
9-10⌹

## 10.6 DIFFERENCE TABLES

The first difference of a vector $Y$ is defined as the vector obtained by taking the difference between each of the pairs of adjacent elements of $Y$. For example, if $Y$ is the vector

```
      0 1 4 9 16 25 36 49 64 81 100
```

then the first difference of $Y$ is the vector

```
      1 3 5 7 9 11 13 15 17 19
```

More precisely, the first difference is the function $D$ defined as follows:

```
      ∇Z←D Y
[1]   Z←(1↓Y)-(¯1↓Y)∇
```

For example:

```
      D Y
1 3 5 7 9 11 13 15 17 19
```

To understand the behavior of the function $D$, it may help to observe the effects of the terms `1↓Y` and `¯1↓Y` as follows:

```
      1↓Y
1 4 9 16 25 36 49 64 81 100

      ¯1↓Y
0 1 4 9 16 25 36 49 64 81
```

The subtraction of the second of these from the first clearly yields the differences between each of the adjacent elements of $Y$.

If `Y←F X` for some function $F$ and some set of equally spaced arguments $X$, then the first difference of $Y$ is also said to be the first difference of the function $F$. For example, if `X←0,⍳10` and `Y←X*2` (that is, $Y$ is the square of $X$), then the vector

```
      D Y
1 3 5 7 9 11 13 15 17 19
```

is said to be the first difference of the square function (for the arguments $X$).

In a function table for $F$, the vectors $X$ and $Y$ used in the preceding paragraph would appear as the first and second columns. Attention will now be limited to function tables whose first column *X* is of the form 0,ι*N*, that is, of the form 0 1 2 3 etc., up to some integer *N*. In the first section of Chapter 11, it will be shown how the methods developed can be applied to any set of equally spaced arguments such as 1.2 1.6 2.0 2.4 2.8 3.2, etc.

Since attention is being confined to argument sets of the form 0,ι*N*, the argument column can be dropped from function tables without introducing ambiguity. For example, the single column on the left of Figure 10.7 shows this simplified form of the function table (for the function *CTOF*) of Table 10.1. The right side of the same figure shows a two-column table containing the function vector *F* and its first difference *D F*; such a table is called a difference table.

| *F* |
|---|
| 32 |
| 33.8 |
| 35.6 |
| 37.4 |
| 39.2 |
| 41 |
| 42.8 |
| 44.6 |
| 46.4 |
| 48.2 |
| 50 |

Abbreviated
Function Table
for Table 10.1

| *F* | *D F* |
|---|---|
| 32 | 1.8 |
| 33.8 | 1.8 |
| 35.6 | 1.8 |
| 37.4 | 1.8 |
| 39.2 | 1.8 |
| 41 | 1.8 |
| 42.8 | 1.8 |
| 44.6 | 1.8 |
| 46.4 | 1.8 |
| 48.2 | 1.8 |
| 50 | 1.8 |

Difference Table
for the Function
*CTOF* of Table 10.1

Function and Difference Table

11-12⍈ Figure 10.7

## 10.7 FITTING FUNCTIONS OF THE FORM *A+B×X*

In using maps to analyze functions, it was found that any function of the form *A+B×X* could be recognized by the uniform spread between adjacent arrow points, and that the actual values of the constants *A* and *B* could be determined from the map. This type of function is analyzed even more easily with the aid of the difference table; the uniform spread is recognized by the fact that the elements of the first difference (which give the spacing between adjacent function values) are all the same. The constants *A* and *B* are simply the first row of the difference table, that is,
13-14⍈ 32 and 1.8 in Figure 10.7.

## 10.8 FACTORIAL POLYNOMIALS

In analyzing certain functions it will be found that the elements of the first difference are not all alike, and the function is therefore not of the form *A+B×X*. In such a case one may take a second difference, that is, the difference of the first difference. If this second difference is not constant, one takes a third difference, and so continues until a constant difference is reached.

For example, Table 10.8 shows a function table in which a constant difference is reached at the third difference.

| *Y* | *D Y* | *D D Y* | *D D D Y* |
|---|---|---|---|
| 5 | ¯2 | 8 | ¯6 |
| 3 | 6 | 2 | ¯6 |
| 9 | 8 | ¯4 | ¯6 |
| 17 | 4 | ¯10 | ¯6 |
| 21 | ¯6 | ¯16 | ¯6 |
| 15 | ¯22 | ¯22 | |
| ¯7 | ¯44 | | |
| ¯51 | | | |

A Constant Third Difference

Table 10.8

The first row of the table is the vector *V*←5 ¯2 8 ¯6. The expression for the function is determined from the vector *V* as follows: *V* is first divided by the vector ! 0 1 2 3 (that is, 1 1 2 6) to obtain the vector *W* as follows:

```
      W←V÷!0 1 2 3
      W
5 ¯2 4 ¯1
```

The elements of *W* are then used to form the following expression:

```
      5+(¯2×X)+(4×X×(X-1))+(¯1×X×(X-1)×(X-2))
```

This expression represents the function exactly, as may be determined by evaluating it for the argument 0,⍳7 and comparing it with the first column of Table 10.8.

The method can be stated in general as follows: Calculate the successive columns of the difference table until a constant column is obtained. Then use the elements of the first row as follows:

Divide the first element by !0 (that is, 1, as shown in Exercise 8.2).
Divide the second element by !1 and multiply by *X*.

Divide the third element by !2 and multiply by X×(X-1).
Divide the fourth element by !3 and multiply by X×(X-1)×(X-2).
and so on.
Finally, add the expressions so obtained.

In other words, if the vector *V* is the first row of the difference table, then the expression

```
(V[I]÷!I-1)× ×/X-¯1+⍳I-1
```

is evaluated for each value of *I* from 1 to ⍴*V*, and the results are then added together. It is important to remember that the foregoing method applies only if the argument column of the function table is of the form 0,⍳*N*, and that the extension to any set of equally spaced arguments will be deferred to Chapter 11.

The functions *X* and *X*×(*X*-1) and *X*×(*X*-1)×(*X*-2), etc., are called <u>factorial polynomials</u>; *X* is called a factorial polynomial of <u>degree</u> 1, and *X*×(*X*-1) is called a factorial polynomial of degree 2, etc. In general, the factorial polynomial of degree *N* is given by the expression ×/*X*-¯1+⍳*N*.

An explanation of why the method works will now be developed. The method is based on the fact that each of the functions *X* and *X*×(*X*-1) and *X*×(*X*-1)×(*X*-2), etc., produce difference tables with particularly simple first rows, and on the fact that difference tables can be added and multiplied by constants in certain useful ways.

Factorial polynomials are important because they can (as illustrated above) be used to fit, and therefore to evaluate, many functions of practical interest. They are also closely related to other important polynomials to be
15-19⊡ introduced in Section 13.5.

## 10.9 MULTIPLICATION AND ADDITION OF DIFFERENCE TABLES

The first difference of a vector has two very useful properties. If *Y* is any vector, if *D Y* is its first difference, and if *A* is any constant, then the first difference of the vector *A*×*Y* is equal to *A* times the first difference of *Y*; that is, *D A*×*Y* is equal to *A*×*D Y*. For example:

```
      Y←0 1 4 9 16 25 36 49
      D Y
1 3 5 7 9 11 13
      6×Y
0 6 24 54 96 150 216 294
      D 6×Y
6 18 30 42 54 66 78
```

```
      6×D Y
6 18 30 42 54 66 78
```

Clearly the same would be true of second differences, third differences, and so on. That is:

```
      |D A×Y          |D D A×Y          |D D D A×Y
      |A×D Y          |A×D D Y          |A×D D D Y
```

Therefore, if every element in a difference table is multiplied by some constant *A*, then it is still a proper difference table, but for the new function *A*×*Y* in its first column.

Similarly, if *Y*1 and *Y*2 are two vectors and if *D Y*1 and *D Y*2 are their first differences, then the first difference of the sum *Y*1+*Y*2 is equal to the sum of the first differences; that is,

```
      D Y1+Y2
      (D Y1)+(D Y2)
```

Again, the same results apply to entire difference tables. Consequently, difference tables may be multiplied by constants and added together at will and the result is always a proper difference table. ⌹20-21

## 10.10 DIFFERENCE TABLES FOR THE FACTORIAL POLYNOMIALS

The factorial polynomials of degrees 0 through 5 are shown below:

| Degree | Polynomial |
|---|---|
| 0 | 1 |
| 1 | *X* |
| 2 | *X*×(*X*-1) |
| 3 | *X*×(*X*-1)×(*X*-2) |
| 4 | *X*×(*X*-1)×(*X*-2)×(*X*-3) |
| 5 | *X*×(*X*-1)×(*X*-2)×(*X*-3)×(*X*-4) |

The polynomial of degree 2 has 2 occurrences of *X*, the polynomial of degree 3 has 3 occurrences of *X*, and so on. The function with a fixed value of 1 has been introduced as the polynomial of degree 0 in order to complete this pattern; it has 0 factors of *X*.

The difference tables for these factorial polynomials are shown in Figure 10.9. Previous tables shown have stopped at the first constant column, but these tables have been continued so that all have the same number of columns. Having the same number of columns, they can be added together. However, it is clear that any columns following a constant column will consist entirely of zeros.

**Degree:** 0
**Function:** 1

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | |
| 1 | 0 | 0 | 0 | | |
| 1 | 0 | 0 | | | |
| 1 | 0 | | | | |
| 1 | | | | | |

**Degree:** 1
**Function:** $X$

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 | |
| 4 | 1 | 0 | 0 | | |
| 5 | 1 | 0 | | | |
| 6 | 1 | | | | |
| 7 | | | | | |

**Degree:** 2
**Function:** $X\times(X-1)$

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 0 | 0 | 0 |
| 0 | 2 | 2 | 0 | 0 | 0 |
| 2 | 4 | 2 | 0 | 0 | 0 |
| 6 | 6 | 2 | 0 | 0 | |
| 12 | 8 | 2 | 0 | | |
| 20 | 10 | 2 | | | |
| 30 | 12 | | | | |
| 42 | | | | | |

**Degree:** 3
**Function:** $X\times(X-1)\times(X-2)$

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 6 | 0 | 0 |
| 0 | 0 | 6 | 6 | 0 | 0 |
| 0 | 6 | 12 | 6 | 0 | 0 |
| 6 | 18 | 18 | 6 | 0 | |
| 24 | 36 | 24 | 6 | | |
| 60 | 60 | 30 | | | |
| 120 | 90 | | | | |
| 210 | | | | | |

**Degree:** 4
**Function:** $X\times(X-1)\times(X-2)\times(X-3)$

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 24 | 0 |
| 0 | 0 | 0 | 24 | 24 | 0 |
| 0 | 0 | 24 | 48 | 24 | 0 |
| 0 | 24 | 72 | 72 | 24 | |
| 24 | 96 | 144 | 96 | | |
| 120 | 240 | 240 | | | |
| 360 | 480 | | | | |
| 840 | | | | | |

**Degree:** 5
**Function:** $X\times(X-1)\times(X-2)$
$\times(X-3)\times(X-4)$

*Y D Y D D Y*

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 120 |
| 0 | 0 | 0 | 0 | 120 | 120 |
| 0 | 0 | 0 | 120 | 240 | 120 |
| 0 | 0 | 120 | 360 | 360 | |
| 0 | 120 | 480 | 720 | | |
| 120 | 600 | 1200 | | | |
| 720 | 1800 | | | | |
| 2520 | | | | | |

The Factorial Polynomials

Figure 10.9

The first row from each table is shown below, together with the degree of the polynomial it is taken from:

| Degree | First Row of Difference Table | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 2 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 6 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 24 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 120 |

Except for final zeros, the first row of the difference table for the factorial polynomial of order *N* is (*N*⍴0),!*N*, that is, *N* zeros followed by !*N*.

Consider now the function obtained as *A* times the zeroth order polynomial added to *B* times the first order polynomial, added to *C* times the second, etc.; that is, the function:

```
A+(B×X)+(C× ×/X-0 1)+(D× ×/X-0 1 2)+(E× ×/X-0 1 2 3)
    +(F× ×/X-0 1 2 3 4)
```

The difference table for this function will be *A* times the difference table for order 0, plus *B* times the difference table for order 1, etc. In particular, the first row of the difference table will be the sum of the following vectors:

```
A × 1   0   0   0   0   0
B × 0   1   0   0   0   0
C × 0   0   2   0   0   0
D × 0   0   0   6   0   0
E × 0   0   0   0  24   0
F × 0   0   0   0   0 120
```

This sum is clearly equal to (*A*,*B*,*C*,*D*,*E*,*F*)×1 1 2 6 24 120, or more simply (*A*,*B*,*C*,*D*,*E*,*F*)×!0,⍳5. Conversely, the values of *A*,*B*,*C*,*D*,*E*,*F* can be determined from the first row *V* of a difference table as follows: *A*,*B*,*C*,*D*,*E*, and *F* are the elements of the vector *V*÷!0,⍳5. This is the rule which was used in Section 10.8. ⌹22

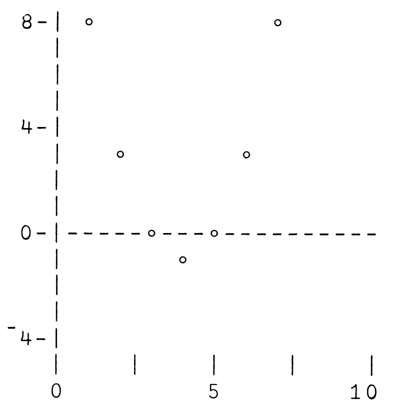
## 10.11 EXPRESSIONS FOR GRAPHS

Consider the function *F* defined and used as follows:

```
      ∇Z←F X
[1]   Z←(X-5)×(X-3)∇

      X←1 2 3 4 5 6 7
      V←F X
      V
8  3  0  ¯1  0  3  8
```

A graph of the function *F* for the arguments *X* is shown in Figure 10.10. The pattern shown by the points of this graph is also shown by the 1's in the following result:

```
      R←8 7 6 5 4 3 2 1 0 ¯1
      R∘.=V
 1 0 0 0 0 0 1
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 1 0 0 0 1 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 1 0 1 0 0
 0 0 0 1 0 0 0
```




Graph of a Parabola

Figure 10.10

The vector *R* is simply the range of the function for the argument *X*, and the comparison between it and the set of values *V* will clearly yield a 1 at each point to be plotted in the graph.

A bar-chart for the same function can be obtained by replacing the comparison for equality by a comparison for less-than-or-equal:

```
      R∘.≤V
 1 0 0 0 0 0 1
 1 0 0 0 0 0 1
 1 0 0 0 0 0 1
 1 0 0 0 0 0 1
 1 0 0 0 0 0 1
 1 1 0 0 0 1 1
 1 1 0 0 0 1 1
 1 1 0 0 0 1 1
 1 1 1 0 1 1 1
 1 1 1 1 1 1 1
```

23-26⍰

The expression $R\circ.=V$ will identify only those elements of $V$ which agree exactly with elements of $R$. For example:

```
      Y←X+.1
      Y
1.1  2.1  3.1  4.1  5.1  6.1  7.1
      W← F Y
      W
7.41  2.61  ¯0.19  ¯0.99  0.21  3.41  8.61
      R∘.=W
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
```

However, one might want to plot points where the agreement is close. This could be done by taking the integer parts of the function values as follows:

```
      ⌊W
7  2  ¯1  ¯1  0  3  8
      R∘.=⌊W
 0 0 0 0 0 0 1
 1 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 1 0
 0 1 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 1 0 0
 0 0 1 1 0 0 0
```

The comparison can also be made as loose or as tight as desired by simply computing the table $|R\circ.-W$ and then comparing it with any desired quantity. For example:

```
      T←|R∘.-W
      T
0.59  5.39  8.19  8.99  7.79  4.59  0.61
0.41  4.39  7.19  7.99  6.79  3.59  1.61
1.41  3.39  6.19  6.99  5.79  2.59  2.61
2.41  2.39  5.19  5.99  4.79  1.59  3.61
3.41  1.39  4.19  4.99  3.79  0.59  4.61
4.41  0.39  3.19  3.99  2.79  0.41  5.61
5.41  0.61  2.19  2.99  1.79  1.41  6.61
6.41  1.61  1.19  1.99  0.79  2.41  7.61
7.41  2.61  0.19  0.99  0.21  3.41  8.61
8.41  3.61  0.81  0.01  1.21  4.41  9.61
```

```
          .5≥T                  1≥T                  2≥T
    0 0 0 0 0 0 0        1 0 0 0 0 0 1        1 0 0 0 0 0 1
    1 0 0 0 0 0 0        1 0 0 0 0 0 0        1 0 0 0 0 0 1
    0 0 0 0 0 0 0        0 0 0 0 0 0 0        1 0 0 0 0 0 0
    0 0 0 0 0 0 0        0 0 0 0 0 0 0        0 0 0 0 0 1 0
    0 0 0 0 0 0 0        0 0 0 0 0 1 0        0 1 0 0 0 1 0
    0 1 0 0 0 1 0        0 1 0 0 0 1 0        0 1 0 0 0 1 0
    0 0 0 0 0 0 0        0 1 0 0 0 0 0        0 1 0 0 1 1 0
    0 0 0 0 0 0 0        0 0 0 0 1 0 0        0 1 1 1 1 0 0
    0 0 1 0 1 0 0        0 0 1 1 1 0 0        0 0 1 1 1 0 0
27⌹ 0 0 0 1 0 0 0        0 0 1 1 0 0 0        0 0 1 1 1 0 0
```

## 10.12 CHARACTER VECTORS

If *P* is a vector of the first five prime integers, then one can index it as shown in the following examples:

```
      P[2]
3
      P[3 1 2]
5  2  3
      P[2 5 4]
3  11  7
      P
2  3  5  7  11
```

Similarly, if *L* is a vector of the first five letters of the alphabet it may be indexed as follows:

```
      L[2]
B
      L[3 1 2]
CAB
      L[2 5 4]
BED
      L
ABCDE
```

The original value of the vector *L* could be assigned by the following expression:

```
      L←'ABCDE'
```

The quotes are necessary to indicate that the result is to be the actual string of characters *ABCDE* rather than some value which has been assigned to the name *ABCDE*. For example:

```
      PRIMES←2 3 5 7 11
      A←PRIMES
      B←'PRIMES'
      A[4 3 2 5]
7  5  3  11
```

```
      B[4 3 2 5]
MIRE
      ⍴A
5
      ⍴B
6
```
⍠28

Characters other than letters can also be used. For example:

```
      C←'*+ABCD'
      C[2 2 1 5 1 3 1 6 1 2 2]
++*C*A*D*++
      ' *'[2 2 1 2 2 1 2 2]
** ** **
```

This last example illustrates how the space may be used as a character. ⍠29

Indexing of a character vector can also be used to display the graphs produced in Section 10.11 in a more pleasing and more readable form. For example, if *R* and *V* are the vectors defined in Section 10.11, then:

```
      R
8 7 6 5 4 3 2 1 0 ¯1
      V
8 3 0 ¯1 0 3 8
      M←R∘.=V
      M
 1 0 0 0 0 0 1
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 1 0 0 0 1 0
 0 0 0 0 0 0 0
 0 0 0 0 0 0 0
 0 0 1 0 1 0 0
 0 0 0 1 0 0 0

      1+M
2 1 1 1 1 1 2
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 2 1 1 1 2 1
1 1 1 1 1 1 1
1 1 1 1 1 1 1
1 1 2 1 2 1 1
1 1 1 2 1 1 1
```

```
      ' *'[1+M]
*     *





 *   *


  * *
   *
```

In order to make such graphing easy we might even define a graphing function *GR* as follows:

```
      ∇Z←GR X
[1]   Z←' *'[1+X]∇

      GR M
*     *





 *   *


  * *
   *

      GR (ι8)∘.≥ι8
*
**
***
****
*****
******
*******
********
```

30-31

# 11

# Inverse Functions

## 11.1 INTRODUCTION

The functions *CTOF* (for Centigrade TO Fahrenheit), and *FTOC*, introduced in Chapter 10, are an example of a pair of mutually inverse functions; that is, *FTOC* undoes the work of *CTOF*, and *CTOF* undoes the work of *FTOC*. This may be stated as follows:

*FTOC CTOF X* yields *X* for any *X*.
*CTOF FTOC X* yields *X* for any *X*.

Examples of the foregoing for particular values of *X* appear in Chapter 10.

Inverse functions are very important. The reason is that whenever one needs to use a certain function, the need for the inverse almost invariably arises. Suppose, for example, that *F* is a function which yields the amount of heat produced by an electric heater as a function of the voltage applied to it. Then for any given voltage *V* one can determine the heat produced by using the expression *F V*. However, if one wants to produce a specified amount of heat *H*, it will be necessary to determine what voltage will produce it. This requires the use of the function inverse to *F* which will yield the voltage as a function of the heat. If this inverse function is called *G*, then the necessary voltage is given by *G H*. Moreover:

*G F X* yields *X* for any *X*.
*F G X* yields *X* for any *X*.

It is therefore important to investigate methods for determining the inverse of any given function *F*. If *F* is represented by a function table, then the inverse function is represented by the same table, but with the argument and function columns interchanged. For example, Table 10.1 (reproduced in the left side of Figure 11.1) represents the function *CTOF* for a certain set of arguments. To apply the function *CTOF* to the argument 3, one locates the value 3 in the first column of the table and then takes the second value in that row (that is, 37.4) as the result. To apply the inverse function *FTOC*, to the argument 41, one locates 41 in the <u>second</u> column and takes the first element in that row (that is, 5) as the result. In other words, the appropriate function table for the inverse function is obtained from the function table for the original function by interchanging the two columns as shown on the right of Figure 11.1.

| *C* | *F* |
|---|---|
| 0 | 32 |
| 1 | 33.8 |
| 2 | 35.6 |
| 3 | 37.4 |
| 4 | 39.2 |
| 5 | 41 |
| 6 | 42.8 |
| 7 | 44.6 |
| 8 | 46.4 |
| 9 | 48.2 |
| 10 | 50 |

| *F* | *C* |
|---|---|
| 32 | 0 |
| 33.8 | 1 |
| 35.6 | 2 |
| 37.4 | 3 |
| 39.2 | 4 |
| 41 | 5 |
| 42.8 | 6 |
| 44.6 | 7 |
| 46.4 | 8 |
| 48.2 | 9 |
| 50 | 10 |

A Pair of Inverse Functions

Figure 11.1

## 11.2 INVERSE OF THE FUNCTION *A*+*B*×*X*

If *F* is the function *A*+*X*, that is:

```
∇Z←F X
Z←A+X ∇
```

then the inverse function is given by *X*-*A* or, equivalently, by (-*A*)+*X*. Thus the inverse function *G* is defined as follows:

```
∇Z←G X
Z←(-A)+X ∇
```

It is easy to see that *F* and *G* are inverse, for *G F X* is equivalent to (-*A*)+*A*+*X* and since (-*A*)+*A* is zero, this is equivalent to 0+*X*, or simply *X* as required. Similarly, *F G X* is equivalent to *A*+(-*A*)+*X* which is equivalent to 0+*X* or *X*.

If *H* is the function *B*×*X*, the inverse function *K* is the function *X*÷*B*, or (÷*B*)×*X*. Thus:

```
∇Z←H X          ∇Z←K X
Z←B×X ∇         Z←(÷B)×X ∇
```

From the foregoing results for addition and multiplication, it should be clear that the inverse of the function $A+B\times X$ is the function $(\div B)\times(-A)+X$. Thus if $L$ and $M$ are defined as follows:

```
∇A←L X                    ∇Z←M X
Z←A+B×X ∇                 Z←(÷B)×(-A)+X ∇
```

then:

```
L M X                     M L X
A+B×(÷B)×(-A)+X           (÷B)×(-A)+A+B×X
A+1×(-A)+X                (÷B)×0+B×X
A+(-A)+X                  (÷B)×B×X
0+X                       1×X
X                         X
```
⌹1-2

## 11.3 DIFFERENCE TABLES

These results will now be applied to extend the applicability of the difference table method of function analysis developed in Chapter 10. Recall that the method developed applies only to a set of arguments of the form 0, 1, 2, 3, etc. Thus the difference table for a function whose values are 4 ¯1 ¯2 1 8 19 would appear as follows if the argument column was added:

| *X* | *Y* | *D Y* | *D D Y* |
|---|---|---|---|
| 0 | 4 | ¯5 | 4 |
| 1 | ¯1 | ¯1 | 4 |
| 2 | ¯2 | 3 | 4 |
| 3 | 1 | 7 | 4 |
| 4 | 8 | 11 | |
| 5 | 19 | | |

The function $F$ represented by the table is obtained by using the first row of the difference table (that is, 4 ¯5 4) divided by the vector 1 1 2 to obtain the coefficients 4 ¯5 2 for the following expression: 4+(¯5×X)+2×X×(X-1). Therefore, the required function $F$ is defined as follows:

```
∇Z←F X
Z←4+(¯5×X)+2×X×(X-1) ∇
```

Evaluation of the expression F 0,⍳5 serves as a check as follows:

```
      F 0 1 2 3 4 5
4 ¯1 ¯2 1 8 19
```

Suppose now that the desired arguments were the equally spaced values *P*←2.0 2.2 2.4 2.6 2.8 3.0. The following table shows these arguments appended to the difference table as a leftmost column:

| *P* | *X* | *Y* | *D Y* | *D D Y* |
|---|---|---|---|---|
| 2 | 0 | 4 | ¯5 | 4 |
| 2.2 | 1 | ¯1 | ¯1 | 4 |
| 2.4 | 2 | ¯2 | 3 | 4 |
| 2.6 | 3 | 1 | 7 | 4 |
| 2.8 | 4 | 8 | 11 | |
| 3 | 5 | 19 | | |

Suppose that one were able to determine a function *G* which yields the column *X* as a function of *P*, that is:

```
      G 2 2.2 2.4 2.6 2.8 3
0 1 2 3 4 5
```

Then *F G P* would yield *Y*; that is:

```
      F G 2 2.2 2.4 2.6 2.8 3
4 ¯1 ¯2 1 8 19
```

In other words, the function *H* defined as follows is the required function:

```
      ∇ Z←H X
      Z←F G X ∇
```

It remains to determine the function *G* which yields the column *X* as a function of the column *P*. Since *X* is of the form 0 1 2 3 4 5, it is easy to determine *P* as a function of *X*, that is, to determine the function <u>inverse</u> to *G*. This is done by forming the difference table for *P* as follows:

| *X* | *P* | *D P* |
|---|---|---|
| 0 | 2 | .2 |
| 1 | 2.2 | .2 |
| 2 | 2.4 | .2 |
| 3 | 2.6 | .2 |
| 4 | 2.8 | .2 |
| 5 | 3 | |

The coefficients 2 .2 in the first row yield the expression 2+.2×*X* for the function inverse to *G*. This is of the form *A*+*B*×*X* and its inverse (that is, *G*) is therefore (÷*B*)×(-*A*)+*X*. Hence *G* is defined as follows:

```
      ∇ Z←G X
      Z←5×¯2+X ∇
```

Finally:

```
      G 2 2.2 2.4 2.6 2.8.3
0 1 2 3 4 5

      F G 2 2.2 2.4 2.6 2.8 3
4 ¯1 ¯2 1 8 19

      H 2 2.2 2.4 2.6 2.8 3
4 ¯1 ¯2 1 8 19
```

Instead of defining and using the separate functions *F* and *G*, their effect could be combined in a single (but cumbersome) expression by substituting for each occurrence of *X* in the expression for *F*, the expression occurring in the function *G*. Thus, for each *X* in the expression

```
      4+(¯5×X)+2×X×(X-1)
```

substitute the expression

```
      5×¯2+X
```

to obtain the single expression

```
      4+(¯5×(5×¯2+X))+2×(5×¯2+X)×((5×¯2+X)-1)
```
⌹3-4

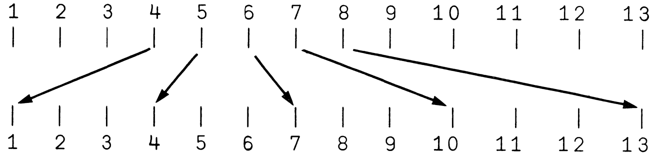
## 11.4 MAPS

In Chapter 10, it was shown how maps and graphs could be useful guides in the analysis of functions. They can also be useful guides in determining inverse functions.

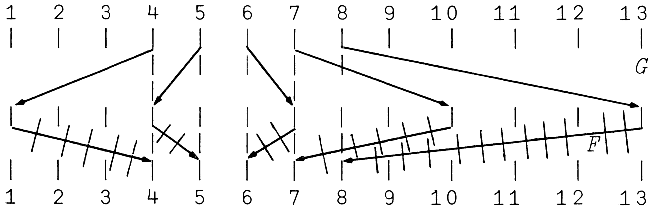
If *F* and *G* are each monadic functions, then we will write *F G* to denote the function defined by applying *F* to the result of *G*. That is, the function *F G* applied to *X* yields *F G X*. If *F* and *G* are inverses, then *F G* must be the <u>identity</u> function, that is, the function which applied to any argument *X* yields *X*.

Consider a function $G$ represented by the following function table and the corresponding map:

| $X$ | $Y$ |
|---|---|
| 4 | 1 |
| 5 | 4 |
| 6 | 7 |
| 7 | 10 |
| 8 | 13 |




A map of the identity function clearly consists of a set of vertical arrows. Therefore, if the identity function is represented by broken line arrows and superimposed on the preceding map, the picture appears as follows:




The function $F$ represented by the crossed lines is clearly the inverse of $G$, since the application of $F$ to the results
5 of $G$ produces the equivalent of the identity function.

## 11.5 GRAPHS

In a graph, the values of the argument $X$ are represented by distances measured along a horizontal line, and the values of the function values $Y$ are represented by distances measured along a vertical line. Since an inverse function is obtained by exchanging the roles of argument and result in the original function, the graph of the inverse is obtained from the graph of the original function by interchanging the horizontal and vertical lines in the graph.

This interchange is easily visualized as follows:

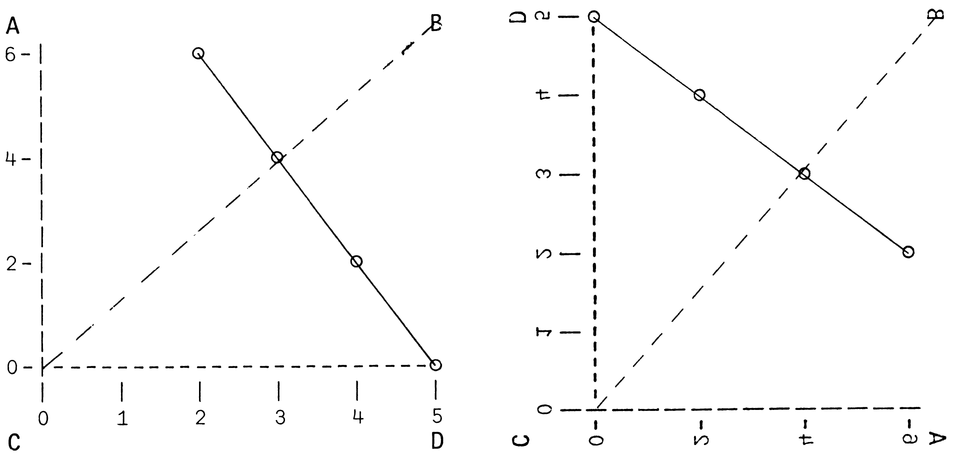
1. Draw the graph of the original function on translucent paper (which can be read through from the obverse side of the paper).

2. Label the top two corners of the paper with *A* and *B*, and the bottom two corners with *C* and *D* (both pairs in order from left to right).

3. Grasp the paper by corners *B* and *C* and flip it over without changing the positions of the two corners held.

The result is a graph of the inverse function.

For example, the left side of Figure 11.2 shows a function table and the corresponding graph. The right side shows the table for the inverse function together with the graph obtained by the process just described. The broken line midway between the *X*-axis and the *Y*-axis shows the line through the points *B C* about which the paper is flipped. It is the one line in the graph whose position remains unchanged.




| *X* | *Y* |
|---|---|
| 2 | 6 |
| 3 | 4 |
| 4 | 2 |
| 5 | 0 |

| *X* | *Y* |
|---|---|
| 6 | 2 |
| 4 | 3 |
| 2 | 4 |
| 0 | 5 |

Inverse Graph by Reflection

Figure 11.2

The graph of an inverse function can, of course, be obtained without using translucent paper, by simply plotting it from the table for the inverse function. One advantage of this is that the scales (the numbers along the horizontal and vertical axes) do not appear lying on their sides and printed backwards as in Figure 11.2. Figure 11.3 shows a pair of functions (the square function $X*2$ and its inverse) in which the graph of the inverse has been drawn in this manner.

```
    |                               o
2.4-|
    |
2.0-|                           o
    |
1.6-|
    |                       o
1.2-|
    |                   o
 .8-|
    |               o
 .4-|           o
    |       o
  0-o------------------------------------------------
    |       |       |       |       |       |       |
    0      .4      .8     1.2     1.6     2.0     2.4
```

```
    |
2.4-|
    |
2.0-|
    |
1.6-|                                                 o
    |                                       o
1.2-|                           o
    |               o
 .8-|
    |       o
 .4-|   o
    |
  0-o----------------------------------------------------
    |       |       |       |       |       |       |
    0      .4      .8     1.2     1.6     2.0     2.4
```

| *X* | *Y* |
|---|---|
| 0 | 0 |
| .2 | .04 |
| .4 | .16 |
| .6 | .36 |
| .8 | .64 |
| 1.0 | 1.00 |
| 1.2 | 1.44 |
| 1.4 | 1.96 |
| 1.6 | 2.56 |

| *X* | *Y* |
|---|---|
| 0 | 0 |
| .04 | .2 |
| .16 | .4 |
| .36 | .6 |
| .64 | .8 |
| 1.00 | 1.0 |
| 1.44 | 1.2 |
| 1.96 | 1.4 |
| 2.56 | 1.6 |

Graphs of a Pair of Inverse Functions

Figure 11.3

The function inverse to the square function is called the square root function. It was treated briefly in Section 6.6 where it was shown that the square root of $X$ is
6-9⌹ equivalent to $X*.5$.

## 11.6 DETERMINING THE INVERSE FOR A SPECIFIC ARGUMENT

For any function whose graph is a straight line, it is easy to find an expression for the function since it is only necessary to determine the values of the constants *A* and *B* in the expression *A*+*B*×*X*. It is equally easy to obtain the expression for the inverse function since this is given by (÷*B*)×(-*A*)+*X*. For example, the function graphed on the left of Figure 11.1 is given by the expression 10+¯2×*X* and the inverse on the right is given by ¯.5×¯10+*X*.

For a function whose graph is not a straight line, it may be impossible to obtain an expression for the inverse function. However, it is possible to determine the inverse function in the following sense: for any given argument in the domain of the inverse function it is possible to determine the corresponding value of the result of the inverse function.

For example, in the case of the square function (*X**2) graphed on the left of Figure 11.3 we have no expression for the inverse function, the square root, graphed on the right. However, for any particular argument it is possible to find the result approximately from the graph of the inverse; for example, if the argument is 2, the result of the inverse function is clearly slightly greater than 1.4. Moreover, one can achieve the same without the graph of the inverse, by working directly from the graph of the original function. Thus one locates the argument 2 on the vertical axis and determines the approximate corresponding result on the horizontal axis.

Finally, one can work directly from the expression for the original function without even graphing it. For example, the expression for the function on the left of Figure 11.2 is *X**2. To obtain the value of the inverse function applied to the argument 2, one must determine a value of *X* such that *X**2 is equal to 2. If one determines a value *C* such that *C**2 is less than 2 and another value *D* such that *D**2 is greater than 2, then the required value of the square root of 2 must lie between *C* and *D*.

Thus, if *C* is 1.4 and *D* is 1.42, then *C**2 is 1.96 and *D**2 is 2.0164 and the required value lies between 1.4 and 1.42. The point midway between them is (1.4+1.42)÷2, that is 1.41. Since 1.41*2 is equal to 1.9881, the required value is greater than 1.41. Since it is already known to be less than 1.42, we now choose the value midway between 1.41 and 1.42, that is, 1.415. The value of 1.415*2 is 2.012225 which is very near to 2. Hence 1.415 is a very good approximation to the value of the square root function applied to the argument 2. Moreover, the same process could be continued to determine better and better approximations as long as desired.

Although we have not obtained an expression for the square root function, we have devised a process which determines the value of the square root when applied to the particular argument 2. Moreover, the process could be applied for any argument other than 2 which lies in the domain of the square root. Finally, the process uses only the expression for the original square function.

The procedure used to determine the square root had to be repeated or *iterated* a number of times to obtain a sufficiently good approximation to the desired result. Such a process is called *iterative*. Functions which are defined by iterative procedures will be discussed more fully in the
10-11 succeeding chapter.

## 11.7 THE SOLUTION OF EQUATIONS

If $G$ is the function inverse to $F$, and one wishes to obtain the value of $G\ N$, then the required value $Y$ must be such that $F\ Y$ is equal to $N$. In other words, the following expression must be true (that is, have the value 1):

$$N=F\ Y$$

Such an expression which is required to be true is called an *equation*, and a value of $Y$ which makes it true is called a *solution* or *root* of the equation.

The problem of determining the value of the inverse function $G$ applied to the argument $N$ is therefore equivalent to finding a solution to the equation $N=F\ Y$. It is for this reason that the solution of equations is a very important topic in the study of algebra. For example, finding the square root of 2 is equivalent to solving the equation $2=X*2$, and finding the square root of 10 is equivalent to solving the equation $10=X*2$.

The origin of the term "square root" for the function inverse to the square function should now be clear; the square root of the argument $N$ is the solution or *root* of the equation $N=X*2$ in which the square function occurs to the
12-13 right of the equal sign.

# 12
# Iterative Processes

## 12.1 INTRODUCTION

The iterative process used in Section 11.6 for finding the square root of 2 is only one of many possible iterative processes for achieving the same end. The following procedure is, in fact, more effective than the procedure of Chapter 11 in the sense that it closes in on the desired value in fewer iterations.

Suppose that $S$ is the square root of a given number $X$, that $Z$ is any other number, and that $Y$ is equal to $X \div Z$. Then $Z \times Y$ is equal to $X$, and $S \times S$ is also equal to $X$. Hence if $Z$ is less than $S$, then $Y$ must be greater than $S$, and if $Z$ is greater than $S$, then $Y$ must be less. In any case, the correct square root $S$ must lie between $Z$ and $Y$. Consequently, the point midway between $Z$ and $Y$ (that is, $.5 \times Z+Y$) should furnish a good new approximation to the square root $S$. Since $Y$ is equal to $X \div Z$, this expression can be written simply as $.5 \times Z+X \div Z$.

Suppose, for example, that we wish to find the square root of 3, that is, $X$ has the value 3. If we choose a value of 1 for $Z$, then the next approximation is given as follows:

```
      X←3
      Z←1
      .5×Z+X÷Z
2
```

Respecifying $Z$ by the new approximation 2 yields a new approximation which can again be used to respecify $Z$:

```
      Z←2
      .5×Z+X÷Z
1.75
      Z←1.75
      .5×Z+X÷Z
1.732142857
      Z←1.732142857
      .5×Z+X÷Z
1.73205081
```

Squaring this last result yields 3.000000008, showing that it is a good approximation to the square root of 3.

The foregoing procedure can be made clearer by simply assigning the value of the new approximation to the name Z each time as follows:

```
      X←3
      Z←1
      Z←.5×Z+X÷Z
      Z
2
      Z←.5×Z+X÷Z
      Z
1.75
      Z←.5×Z+X÷Z
      Z
1.732142857
      Z←.5×Z+X÷Z
      Z
1.73205081
```

From this it is clear that the *iteration* consists of repeating the execution of the expression `Z←.5×Z+X÷Z` enough times, the line containing only the expression `Z` being inserted solely to allow us to see the successive values of the approximation `Z`.

Such iteration can be specified in a function definition as follows:

```
      ∇Z←SQRT X
[1]   Z←1
[2]   Z←.5×Z+X÷Z
[3]   →2∇
```

The right-pointing arrow on line 3 of the function definition is called a *branch*; the only effect of the expression `→2` is to cause statement number 2 to be executed next. Hence statements 2 and 3 are executed again and again in sequence. This behavior can be seen from a trace of the function as follows:

```
      T∆SQRT←1 2 3
      P←SQRT 3
SQRT[1] 1
SQRT[2] 2
SQRT[3] 2
SQRT[2] 1.75
SQRT[3] 2
SQRT[2] 1.732142857
SQRT[3] 2
SQRT[2] 1.73205081
```

The trouble with the function SQRT is that it never terminates. It would be desirable to make it terminate when a certain condition becomes satisfied, say when the magnitude of the difference between `Z*2` and the argument X becomes less than .00001. This is achieved in the function *SQT* defined as follows:

```
      ∇Z←SQT X
[1]    Z←1
[2]    Z←.5×Z+X÷Z
[3]    →2×.00001<|X-Z*2∇
```

As long as *X* and *Z**2 differ by .00001 or more, the expression following the branch arrow is equal to 2×1 and statement 2 is executed next. When *Z**2 becomes close enough to *X*, the expression has the value 2×0, (that is, 0), indicating that statement 0 should be executed next. Since there is no statement 0, the process terminates.

The function *SQT* can now be applied to any non-negative argument. For example:

```
      SQT 2
1.4142156862745
      (SQT 2)*2
2.0000060073049
      SQT 10
3.1622776651757
      (SQT 10)*2
10.000000031668
```

The detailed behavior of the function *SQT* can be seen in a trace as follows:

```
      T∆SQT←1 2 3
      P←SQT 10
SQT[1] 1
SQT[2] 5.5
SQT[3] 2
SQT[2] 3.6590909090909
SQT[3] 2
SQT[2] 3.1960050818746
SQT[3] 2
SQT[2] 3.1624556228039
SQT[3] 2
SQT[2] 3.1622776651757
SQT[3] 0
      P
3.1622776651757
```

Iteration is of great importance in mathematics and its uses are by no means limited to root-finding. The remaining sections of this chapter illustrate a few of its uses. Others occur in later chapters. ⌹2

## 12.2 GENERAL ROOT FINDER

The iterative method used in Section 11.6 to determine the square root of 2 can now be expressed as a formal function definition by using branching. The method consists of using two quantities *C* and *D* which bound the desired value in the following sense: *C*⋆2 is less than 2 and *D*⋆2 is greater than 2, and the desired value therefore lies between *C* and *D*. The method procedes by computing the point *Z* midway between *C* and *D* and then computing *Z*⋆2 to see whether it lies above or below 2. If it lies below 2, then *C* is respecified by *Z* (that is, *C*←*Z*) and the process is repeated; otherwise *D* is respecified by *Z* and the process is repeated.

It will be more convenient to combine the bounding quantities *C* and *D* in a single vector *B* so that *Z* respecifies either *B*[1] or *B*[2]. The complete definition and traced behaviour of the function follow:

```
      ∇Z←Q X
[1]   B←1.4 1.42
[2]   Z←.5×+/B
[3]   I←1+X<Z⋆2
[4]   B[I]←Z
[5]   →2×.00001<|X-Z⋆2∇

      TΔQ←⍳5
      P←Q 2
Q[1] 1.4  1.42
Q[2] 1.41
Q[3] 1
Q[4] 1.41
Q[5] 2
Q[2] 1.415
Q[3] 2
Q[4] 1.415
Q[5] 2
Q[2] 1.4125
Q[3] 1
Q[4] 1.4125
Q[5] 2
Q[2] 1.41375
Q[3] 1
Q[4] 1.41375
Q[5] 2
Q[2] 1.414375
Q[3] 2
Q[4] 1.414375
Q[5] 2
Q[2] 1.4140625
Q[3] 1
Q[4] 1.4140625
Q[5] 2
Q[2] 1.41421875
Q[3] 2
Q[4] 1.41421875
Q[5] 2
Q[2] 1.414140625
Q[3] 1
Q[4] 1.414140625
Q[5] 2
Q[2] 1.4141796875
Q[3] 1
Q[4] 1.4141796875
Q[5] 2
Q[2] 1.41419921875
Q[3] 1
Q[4] 1.41419921875
Q[5] 2
Q[2] 1.414208984375
Q[3] 1
Q[4] 1.414208984375
Q[5] 2
Q[2] 1.4142138671875
Q[3] 2
Q[4] 1.4142138671875
Q[5] 0
      P
1.4142138671875
```

The foregoing function will determine a root of the equation *X*=*Z*⋆2, that is, for a given value of *X* it will determine a value of *Z* such that the equation is true. In order to obtain a general root finder which would solve the equation *X*=*F Z* for any desired function *F*, it is necessary to replace every occurrence of the expression *Z*⋆2 in the function *Q* by the expression *F Z*.

It will also be convenient to have the bounding vector *B* as an argument of the function so that one can specify suitable initial bounding values. The general root-finder is therefore defined as follows:

```
      ∇Z←B GRF X
[1]   Z←.5×+/B
[2]   B[1+X<F Z]←Z
[3]   →.00001<|X-F Z∇
```

Suppose, for example, that *F* is the cube function defined as follows:

```
      ∇Z←F X
[1]   Z←X⋆3∇
```

Then, since 4⋆3 is less than 100 and 5⋆3 is greater than 100, the expression 4 5 *GRF* 100 yields a solution of the equation 100=*Z*⋆3 as follows:

```
      4 5 GRF 100
4.6415886878967
      (4 5 GRF 100)⋆3
99.999990581929
```
⌹3

There are two reasons for including the bounding values *B* as an argument of the general root finder function *GRF*. The first is that for some functions *F* it is very difficult to compute suitable initial bounding values and it may be necessary to provide them, possibly from information obtained from a rough graph. The second reason is that for some functions *F* the equation *X*=*F Z* has more than one solution, and the initial bounding values permit us to isolate any one of the several roots as desired.

For example, suppose that *F* is defined as follows:

```
      ∇ Z←F X
        Z←¯76.44+(102.2×X)+(¯41×X⋆2)+(5×X⋆3) ∇
```

Then several different values of *X* can be determined for which *F X* is zero:

```
      1 2 GRF 0
1.4
      3 2 GRF 0
2.6
      4 5 GRF 0
4.2
```

It can be verified that this function is equivalent to the function 5×(X-1.4)×(X-2.6)×(X-4.2) whose graph appears in Figure 10.5. This graph will therefore be helpful in appreciating how the different bounding values lead to different roots. Two further solutions appear below:

```
      1 2 GRF 3
1.64639
      3 2 GRF 3
2.23409
```

4-6⌹

## 12.3 GREATEST COMMON DIVISOR

A common divisor of two integers is an integer which is a factor of both, and the greatest common divisor is the largest of such common divisors. For example, the numbers 24 and 54 have the following divisors:

```
      (0=(⍳24)|24)/⍳24
1  2  3  4  6  8  12  24
      (0=(⍳54)|54)/⍳54
1  2  3  6  9  18  27  54
```

The common divisors are 1 2 3 6, and the greatest common divisor is therefore 6.

An interesting and efficient method for finding the greatest common divisor of a pair of integers *X* and *Y* is based on the following fact: If *Z* is the remainder obtained on dividing *X* into *Y* (that is, Z←X|Y), then the greatest common divisor of *X* and *Y* is also the greatest common divisor of *X* and *Z*. For example, if *X* is 48 and *Y* is 66, then *Z* is 18 and the greatest common divisor of 48 and 66 is the same as the greatest common divisor of 18 and 48. The process can now be repeated since the the greatest common divisor of 18 and 48 is the greatest common divisor of 18 and their remainder, which is 12. Thus we look for the greatest common divisor of 12 and 18. The remainder 12|18 is 6 and we now look at the pair 6 and 12. The remainder 6|12 is zero. This indicates that 6 is a divisor of 12 and therefore 6 is the greatest common divisor of 6 and 12. Hence, 6 is also the greatest common divisor of the original pair 48 and 66.

The foregoing is an iterative process which can obviously be defined as follows:

```
      ∇Z←X GD Y
[1]   Z←X
[2]   X←X|Y
[3]   Y←Z
[4]   →X≠0∇
```

The behavior of the function *GD* can be seen from the following trace:

```
      T∆GD←⍳4
      P←48 GD 66
GD[1] 48
GD[2] 18
GD[3] 48
GD[4] 1
GD[1] 18
GD[2] 12
GD[3] 18
GD[4] 1
GD[1] 12
GD[2] 6
GD[3] 12
GD[4] 1
GD[1] 6
GD[2] 0
GD[3] 6
GD[4] 0
      P
6
```

The greatest common divisor function can also be defined in terms of a single argument (which is expected to be a two-element vector) as follows:

```
     ∇ Z←GCD X
[1]   Z←X[1]
[2]   X←(|/X),X[1]
[3]   →X[1]≠0 ∇
```

For example:

```
      T∆GCD←⍳3
      P←GCD 48 66
GCD[1] 48
GCD[2] 18 48
GCD[3] 1
GCD[1] 18
GCD[2] 12 18
GCD[3] 1
GCD[1] 12
GCD[2] 6 12
GCD[3] 1
GCD[1] 6
GCD[2] 0 6
GCD[3] 0
      P
```
7⍠ 6

The function *GCD* can be used in the treatment of rational numbers as follows. If *V* is any two-element vector of integers it can be used to represent the rational number ÷/*V*. Moreover, if *V* is multiplied by any scalar integer *S* it still represents the same rational number. For example:

```
      V←48 66
      ÷/V
0.727273
      3×V
144 198
      ÷/3×V
0.727273
```

Similarly, if *V* is divided by any integer which is a divisor of both elements, the result is a pair of integers which also represent the same rational number. For example:

```
      V÷2
24 33
      ÷/V÷2
0.727273
```

Moreover, if *V* is divided by the greatest common divisor of *V*[1] and *V*[2], one obtains the smallest pair of integers which represent the same rational. For example:

```
      V÷GCD V
8 11
      ÷/V÷GCD V
```
8-11⍠ 0.727273

## 12.4 THE BINOMIAL COEFFICIENTS

Binomial coefficients are of importance in many areas of mathematics. In this section they will be introduced as a further example of the use of iteration in the function which defines them. They will be used and studied more thoroughly in Section 15.5.

The binomial coefficients of order *N* are the *N*+1 elements of the vector produced by the expression *BIN N* using the function *BIN* defined as follows:

```
    ∇Z←BIN X
[1]   Z←,1
[2]   →3×X≥ρZ
[3]   Z←(Z,0)+(0,Z)
[4]   →2 ∇
```

The following examples illustrate the behavior of the function:

```
      BIN 0
1
      BIN 1
1 1
      BIN 2
1 2 1
      BIN 3
1 3 3 1
      BIN 4
1 4 6 4 1
      BIN 5
1 5 10 10 5 1
      BIN 6
1 6 15 20 15 6 1

      TΔBIN←ι4
      P←BIN 3
BIN[1] 1
BIN[2] 3
BIN[3] 1 1
BIN[4] 2
BIN[2] 3
BIN[3] 1 2 1
BIN[4] 2
BIN[2] 3
BIN[3] 1 3 3 1
BIN[4] 2
BIN[2] 0
      P
1 3 3 1
```

⌹12-19

# 13

# Inner Products and Polynomials

## 13.1 INTRODUCTION

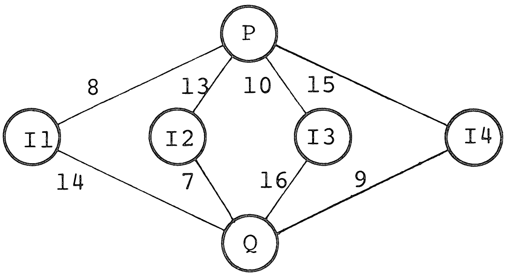
Each of the expressions $+/D\times W$ and $\lfloor/A+B$ and $\lceil/A\lfloor B$ involve a dyadic function applied to the two arguments, followed by a reduction of this result by a second dyadic function applied over the result. These expressions are therefore said to be of the same form, although they do differ in the actual dyadic functions employed. Thus the first uses $+$ and $\times$, the second uses $\lfloor$ and $+$, and the third uses $\lceil$ and $\lfloor$.

Expressions of this form are so important that they will be assigned a special notation known as inner product. Their importance is due largely to the fact that they arise very frequently in practical problems. Consider, for example, the following expressions:

```
      D←5 2 4
      W←36 12 1
      +/D×W
208
      A←8 13 10 15
      B←14 7 16 9
      ⌊/A+B
20
```

The expression $+/D\times W$ may arise from a practical problem as follows. Suppose that the elements of $D$ express a certain distance in terms of yards, feet, and inches, that is, $D$ represents the distance 5 yards, 2 feet, and 4 inches. One could express tne same distance in inches alone by multiplying the first element by 36, the second by 12, the third by 1, and then summing the results. In other words, if $W$ is the weighting vector as specified above, then the distance in inches is given by the expression $+/D\times W$.

The second expression $\lfloor/A+B$ may arise as follows. Suppose that one wishes to travel from station $P$ to station $Q$ and has a choice of four different routes, via the four different intermediate stations, $I1$, $I2$, $I3$, and $I4$ as shown in Figure 13.1. Suppose further that the distances from $P$ to the four intermediate stations are given by the four elements of the vector *A*, and that the distances from the intermediate stations to the destination *Q* are given by the vector *B*. Then the expression *A*+*B* gives the total distances for each of the four possible routes, and ⌊/*A*+*B* gives the smallest of these distances, that is, the shortest distance possible by the available routes.




Minimum Distance

Figure 13.1 ⌹1-2

## 13.2 THE INNER PRODUCT OF TWO VECTORS

If *X* and *Y* are vectors of the same dimension, then the expression *X*+.×*Y* is called the plus times inner product of *X* and *Y*, and is defined to be equivalent to the expression +/*X*×*Y*. Similarly, *X*⌊.+*Y* is called the minimum plus inner product and is defined as ⌊/*X*+*Y*, and so on for every pair of dyadic functions. For example:

```
      X←2 3 5 7 11
      Y←2 1 2 0 1
      X+.×Y                        +/X×Y
28                           28
      X⌊.+Y                        ⌊/X+Y
4                            4
      X×.*Y                        ×/X*Y
3300                         3300
      X+.-Y                        +/X-Y
22                           22
      X+.≠Y                        +/X≠Y
4                            4
      X⌈.=Y                        ⌈/X=Y
1                            1
```

⌹3-5

## 13.3 MATRICES

What we have been calling a table is in mathematics more usually called a matrix; we will call it so from now on. We will also generalize the dyadic repetition function (introduced in Section 1.7 and denoted by ⍴) so that it will permit the specification of a matrix with any shape and having any desired elements.

The dyadic repetition function ⍴ was defined only for scalar arguments, but it will now be defined for vector arguments as well. For example:

```
      3⍴5
5  5  5
      5⍴3
3  3  3  3  3
      3⍴1 2 3 4
1  2  3
      10⍴1 2 3 4
1  2  3  4  1  2  3  4  1  2
```

From these examples it is clear that the left argument determines the size of the result and that the elements of the result are chosen from the right argument, repeating them over and over if necessary.

If the left argument *A* is a two-element vector it again determines the size of the result, that is, the result is a matrix *M* such that ⍴*M* (that is, the size of *M*) is equal to *A*. In other words, *M* has *A*[1] rows and *A*[2] columns. For example:

```
      2 3⍴1 2 3 4 5 6
1  2  3
4  5  6

      3 4⍴⍳12
   1   2   3   4
   5   6   7   8
   9  10  11  12

      3 5⍴0 1
 0 1 0 1 0
 1 0 1 0 1
 0 1 0 1 0
```

The expression *V*∘.+*W* has been referred to as an addition table, *V*∘.×*W* has been referred to as a multiplication table, and so on. In advanced mathematics such an expression is more usually referred to as an outer product (outer product for addition, outer product for multiplication, etc.) and we will now adopt this

6-7⌹ terminology.

## 13.4 INNER PRODUCT WITH MATRIX ARGUMENTS

The inner product also applies to matrix arguments. For example:

```
      M←3 4⍴3 0 4 2 4 6 5 1 0 5 2 4
      N←4 5⍴6 7 2 1 7 5 6 5 0 5 7 2 3 6 3 1 2 2 1 3
      M
3  0  4  2
4  6  5  1
0  5  2  4
      N
6  7  2  1  7
5  6  5  0  5
7  2  3  6  3
1  2  2  1  3

      M+.×N                            M⌊.+N
48  33  22  29  39             3  4  4  0  5
90  76  55  35  76             2  3  3  2  4
43  42  39  16 43              5  4  2  1  5

      M+.=N                            M+.≠N
0  1  1  1  0                  4  3  3  3  4
1  1  0  1  0                  3  3  4  3  4
1  1  1  0  1                  3  3  3  4  3

      (M+.=N)+(M+.≠N)
4  4  4  4  4
4  4  4  4  4
4  4  4  4  4
```

The result of an inner product applied to matrices *M* and *N* is a matrix having as many rows as the first argument and as many columns as the second argument. The elements of the results are the results obtained by applying the inner product to each row vector of the first argument paired with each column vector of the second argument. More specifically, if *R*←*M*+.×*N*, then the element *R*[*I*;*J*] is given by the expression *M*[*I*;]+.×*N*[;*J*]. For example:

```
      R←M+.×N
      R
48  33  22  29  39
90  76  55  35  76
43  42  39  16  43

      R[2;3]
55
      M[2;]
4  6  5  1
      N[;3]                              (M⌊.+N)[3;5]
2  5  3  2                         5
      M[2;]+.×N[;3]                      (M[3;]⌊.+N[;5]
55                                 5
```

⌹8-10

If `X` is a vector and `M` is a matrix, then the inner product `M+.×X` is defined by simply treating `X` much like a 1-column matrix. For example:

```
      X←0 3 2 4
      M+.×X
16  32  35
      M⌊.+X
3  4  0
      M+.≠X
4  4  1
      M+.=X
0  0  3
```

If `Y` is a vector and `M` is a matrix, then the inner product `Y+.×M` is defined by treating `Y` much like a 1-row matrix. For example:

```
      Y←0 4 2
      Y+.×M
16  34  24  12
      Y⌊.+M
2  0  4  2
      Y+.≠M
```

11-18⌹
```
2  2  2  3
```

## 13.5 POLYNOMIALS

If `C` is a vector and `X` is a scalar, then an expression of the form `+/C×X*¯1+⍳⍴C` is a function of `X` which is called a <u>polynomial of degree</u> `¯1+⍴C`. For example, if `C←2 5 ¯3 1`, then `+/C×X*¯1+⍳⍴C` is a polynomial of degree 3 and is equivalent to the expression `+/2 5 ¯3 1×X*0 1 2 3`. This expression is clearly equal to the sum of the following quantities:

```
      2×X*0
      5×X*1
      ¯3×X*2
      1×X*3
```

Each of these quantities is called a <u>term</u> of the polynomial; each of the constant multipliers is called a <u>coefficient</u>.

Figure 13.2 shows a graph of each of the terms of the polynomial `+/2 5 ¯3 1×X*0 1 2 3`, together with a graph of their sum, that is, of the polynomial itself.

The polynomial function described above can be defined formally by the following function:

```
      ∇Z←C POLY X
[1]   Z←+/C×X*¯1+⍳⍴,C∇
```

For example:

```
      2 5 ¯3 1 POLY 2
8
      6 POLY 2
6
```

The comma following the `ρ` in the definition of the function *POLY* is included to ravel the argument *C* so as to avoid difficulty in the case of a scalar argument *C* (as occurs in the second use of *POLY* above). (See Sections 8.7 and 8.8.)

Since a polynomial may have any number of terms and since each of the coefficients may have any value, the graphs of Figure 13.2 suggest (correctly) that coefficients can be chosen so as to make a polynomial which approximates any function of practical interest. This ability to approximate a wide variety of functions is one of the main reasons for the overwhelming importance of polynomials. A second reason is the ease of evaluation, which involves only addition, multiplication, and powers. A third reason is the ease with which polynomial functions can be analyzed. ⌹19-21

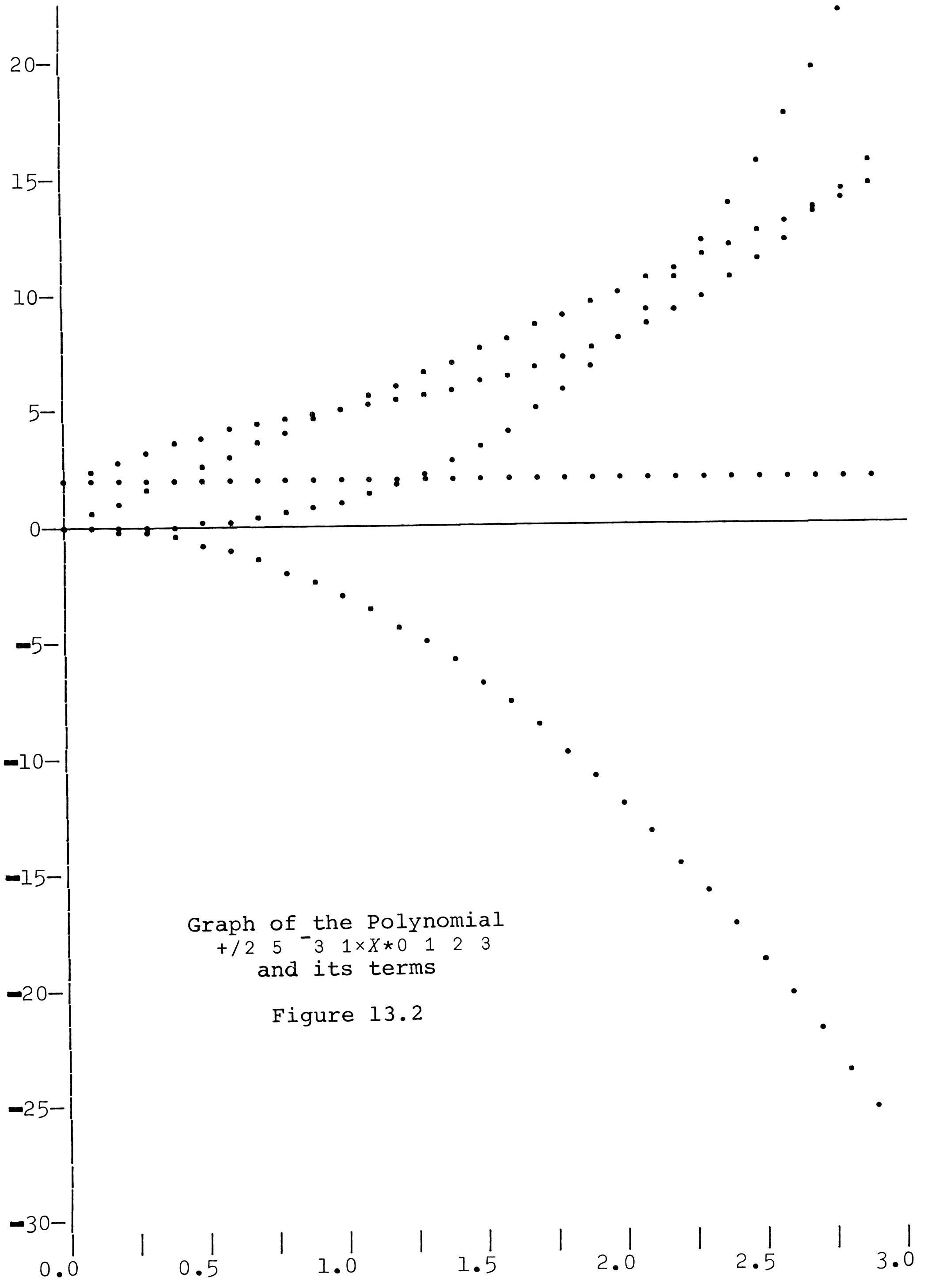
## 13.6 POLYNOMIALS EXPRESSED AS INNER PRODUCTS

Since `P×Q` is equivalent to `Q×P`, the expression `+/C×(X*¯1+⍳ρ,C)` for a polynomial can also be written as `+/(X*¯1+⍳ρ,C)×C`. Moreover, since `+/Q×P` can be written in the inner product form as `Q+.×P`, the polynomial can be written as the inner product `(X*¯1+⍳ρ,C)+.×C`.

It should be clear that none of these equivalent expressions for a polynomial apply correctly to a <u>vector</u> argument *X* in order to evaluate the polynomial applied separately to each element of *X*. For example:

```
      C←1 2 1
      X←3
      +/C×X*¯1+⍳ρ,C
16
      X←4
      +/C×X*¯1+⍳ρ,C
25
      X←5
      +/C×X*¯1+⍳ρ,C
36
      X←3 4 5
      +/C×X*¯1+⍳ρ,C
34
      X←3 4
      +/C×X*¯1+⍳ρ,C
```

(cannot be evaluated because the vectors *X* and `¯1+⍳ρ,C` are not of the same size)




Graph of the Polynomial
+/2 5 ¯3 1×*X**0 1 2 3
and its terms

Figure 13.2

To obtain the correct result of 16 25 36 when applying the polynomial with coefficients C←1 2 1 to the vector argument 3 4 5, it is necessary to use a different expression for the polynomial. This can be obtained by a slight modification of the inner product expression (X*¯1+ιρ,C)+.×C, namely, (X∘.*¯1+ιρ,C)+.×C. For example:

```
      C←1 2 1
      X←3 4 5
      X∘.*¯1+ιρ,C
1   3   9
1   4  16
1   5  25
      (X∘.*¯1+ιρ,C)+.×C
16  25  36
```

The following definition will therefore be adopted for the polynomial function:

```
      ∇Z←C P X
      Z←(X∘.*¯1+ιρ,C)+.×C∇
```

The following examples illustrate its use:

```
      1 2 1 P 3 4 5 6
16  25  36  49
      1 3 3 1 P 3 4 5 6
64  125  216  343
      7 P 3 4 5 6
7  7  7  7
```

⌹22-24

# 14

# Identities

## 14.1 INTRODUCTION

Two expressions are said to be equivalent if they represent the same function, that is, if they both yield the same value for any specified argument (lying within their domains). For example, $X\times Y$ and $Y\times X$ are equivalent, as are $X\lceil Y$ and $Y\lceil X$, but $X-Y$ and $Y-X$ are not equivalent.

If two equivalent expressions are joined by an equal sign, the resulting single expression is true (i.e., has the value 1) for every possible value of the argument or arguments. It is therefore called an identity. For example, the expression $(X\times Y)=(Y\times X)$ is always true, as are $(X\lceil Y)=(Y\lceil X)$ and $(X\lfloor (Y\lfloor Z))=((X\lfloor Y)\lfloor Z)$.

For convenience in discussion, many of the more useful identities are given names. For example, the identity $(X\times Y)=(Y\times X)$ is said to express the commutativity of times, and $(X\lfloor (Y\lfloor Z))=((X\lfloor Y)\lfloor Z)$ expresses the associativity of minimum. The following list shows (together with their names) a number of identities which the reader should either find already familiar, or be able to verify by evaluating them for a few sample values of the arguments:

| Identity | Name |
|---|---|
| $(X+Y)=(Y+X)$ | Commutativity of plus |
| $((X\lceil Y)\lceil Z)=(X\lceil (Y\lceil X))$ | Associativity of maximum |
| $(X\times (Y+Z))=((X\times Y)+(X\times Z))$ | Distributivity of times over plus |
| $(X\lceil (Y\lfloor Z))=((X\lceil Y)\lfloor (X\lceil Z))$ | Distributivity of maximum over minimum |
| $(X\lceil Y)=(-(-X)\lfloor (-Y))$<br>$(X\lfloor Y)=(-(-X)\lceil (-Y))$ | Duality of maximum and minimum |
| $(X\vee Y)=(\sim(\sim X)\wedge(\sim Y))$<br>$(X\wedge Y)=(\sim(\sim X)\vee(\sim Y))$ | Duality of and and or |

Identities are very useful in mathematics, primarily because they allow one to easily express the same function in a variety of ways, each of the different ways possessing some particular advantage such as being easy to evaluate, or providing some particular insight into the behavior of the function. Consider, for example, the function `+/(⍳X)*2` which yields the sum of the squares of the integers up to and including *X*. The difference table for this function appears as follows:

| X | +/(⍳X)*2 | D +/(⍳X)*2 | D D +/(⍳X)*2 | D D D+/(⍳X)*2 |
|---|---|---|---|---|
| 0 | 0 | 1 | 3 | 2 |
| 1 | 1 | 4 | 5 | 2 |
| 2 | 5 | 9 | 7 | 2 |
| 3 | 14 | 16 | 9 | 2 |
| 4 | 30 | 25 | 11 | |
| 5 | 55 | 36 | | |
| 6 | 91 | | | |

According to the method of analyzing a function by difference tables developed in Chapter 10, the first row of the difference table (that is, `0 1 3 2`) can be divided by `!0 1 2 3` (that is, `1 1 2 6`) to obtain the coefficients `0`, `1`, `3÷2`, and `2÷6` used in the following expression:

```
0+X+((3÷2)×X×(X-1))+(2÷6)×X×(X-1)×X-2
```

The expression is equivalent to `+/(⍳X)*2`. Moreover, for large values of *X* it is much easier to evaluate than `+/(⍳X)*2`. For example, the sum of the squares up to `100` is given by:

```
0+100+((3÷2)×100×99)+(2÷6)×100×99×98
0+100+14850+323400
338350
```

Moreover, by methods to be developed in this chapter, the expression `0+X+((3÷2)×X×X-1)+(2÷6)×X×(X-1)×X-2` can be shown to be equivalent to the polynomial:

```
(÷6)×(X*0 1 2 3)+.×0 1 3 2
```

This can be evaluated even more easily. For example:

```
X←100
(÷6)×(X*0 1 2 3)+.×0 1 3 2
(÷6)×1 100 10000 1000000+.×0 1 3 2
(÷6)×0+100+30000+2000000
(÷6)×2030100
338350
```

An argument or series of arguments which clearly establishes the equivalence of two expressions is said to be a <u>proof</u> of their identity. This chapter is intended not only to establish certain important identities, but also to illustrate <u>methods</u> of proof which the reader may use to establish further identities. Of the basic methods of proof, one (mathematical induction) is deferred to Chapter 15.

## 14.2 COMMUTATIVITY

Since *X*+*Y* yields the same result as *Y*+*X*, the function + is said to commute, or to be commutative. The word commute implies that the two arguments can be commuted (i.e., interchanged) without changing the result. The function × is also commutative; that is, (*X*×*Y*)=(*Y*×*X*). To see why this is so, consider the way in which multiplication is defined as repeated addition, that is, 3×4 can be considered as the addition of three groups of objects each containing four items.

This can be pictured in terms of the array

```
      3 4 ρ '⎕'
⎕⎕⎕⎕
⎕⎕⎕⎕
⎕⎕⎕⎕
```

which consists of three rows, each containing four boxes. The total number of boxes is then 3×4. It is clear that the array

```
      ⍉3 4 ρ '⎕'
⎕⎕⎕
⎕⎕⎕
⎕⎕⎕
⎕⎕⎕
```

contains the same number of boxes. It is equally clear that this is the same array as

```
      4 3 ρ '⎕'
⎕⎕⎕
⎕⎕⎕
⎕⎕⎕
⎕⎕⎕
```

which represents the product 4×3. Hence, (3×4)=(4×3).

The functions maximum and minimum are both commutative, that is,

```
      (X⌈Y)=(Y⌈X)
```

and

```
      (X⌊Y)=(Y⌊X)
```

It is equally clear that equality is commutative, that is,

```
      (X=Y)=(Y=X).
```

To show that a function is not commutative, it is sufficient to exhibit one pair of arguments for which it does not commute. For example, 4-3 yields 1 and 3-4 yields ¯1. Since these results differ, it is clear that subtraction is not commutative. Similarly 3≤4 yields 1 and 4≤3 yields 0 and the function ≤ therefore does not commute.

The results thus far can be summarized in a table as follows:

| + | - | × | ⌈ | ⌊ | ≤ | = |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |

A zero lying below a function symbol indicates that the function is not commutative, and a 1 indicates that it is. ⌹1

The 1's and 0's in the foregoing table can be thought of as the results of a function *COM* which determines the commutativity of its argument, that is, *COM* '+' yields 1, and *COM* '-' yilelds 0, and so on. This function could be defined as follows:

```
∇Z←COM X
Z←(X='+-×⌈⌊≤=')/1 0 1 1 1 0 1
```

For example, in the evaluation of the expression *COM* '⌈', the argument *X* has the value '⌈', and the expression *X*='+-×⌈⌊≤=' therefore has the value 0 0 0 1 0 0 0. Consequently, (*X*='+-×⌈⌊≤=')/1 0 1 1 1 0 1 yields 1, indicating that the function maximum is commutative. ⌹2

<u>Function Tables</u>. Consider the subtraction table *S* and its transpose *T*←⍉*S* shown in Figure 14.1. The circled element in *S* is the result of the subtraction 5-3. The corresponding element of *T* (enclosed in a rectangle) is clearly the result of 3-5. More generally, if one uses table *S* to evaluate any subtraction *X*-*Y*, then the corresponding element of table *T* is the result of the commuted expression *Y*-*X*. Consequently, a function is commutative only if its function table *A* agrees with its transpose ⍉*A*.

*S*

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 | ¯6 |
| 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 | ¯5 |
| 2 | 1 | 0 | ¯1 | ¯2 | ¯3 | ¯4 |
| 3 | 2 | 1 | 0 | ¯1 | ¯2 | ¯3 |
| 4 | 3 | 2 | 1 | 0 | ¯1 | ¯2 |
| 5 | 4 | 3 | 2 | 1 | 0 | ¯1 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 |

*S*←(⍳7)∘.-⍳7

*T*

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ¯1 | 0 | 1 | 2 | 3 | 4 | 5 |
| ¯2 | ¯1 | 0 | 1 | 2 | 3 | 4 |
| ¯3 | ¯2 | ¯1 | 0 | 1 | 2 | 3 |
| ¯4 | ¯3 | $\boxed{¯2}$ | ¯1 | 0 | 1 | 2 |
| ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 | 1 |
| ¯6 | ¯5 | ¯4 | ¯3 | ¯2 | ¯1 | 0 |

*T*←⍉*S*

Function Tables for Subtraction

Figure 14.1

Most functions of interest are defined on a limitless domain (e.g., all numbers) and any function table therefore represents only a part of the domain. Consequently, the fact that a function table agrees with its transpose does not prove that the function is commutative, since an enlarged table might show that it is not. However, some important functions are defined for a limited domain (i.e., for only a small number of argument values), and for such a function it is possible to make a complete function table and determine the properties of the function directly from the table.

We will illustrate this by defining four important *logical* functions, i.e., functions whose ranges are limited to the logical values 0 and 1. They are called *and*, *or*, *not-and*, and *not-or*, and are denoted by ∧, ∨, ⍲, and ⍱, respectively. They are completely defined by the function tables of Figure 14.2. These tables are all symmetric (i.e., agree with their transposes), and these functions are therefore all commutative.

| ∧ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

and

| ∨ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

or

| ⍲ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 0 |

not-and

| ⍱ | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 0 |

not-or

Function Tables for Logical Functions

3⊟ Figure 14.2

*The Method of Exhaustion*. The process of examining all possible cases to determine some property of a function (used above on the logical functions) is called the *method of exhaustion*. It can often be applied even if the number of possible values of the arguments is unlimited. For example, the arguments of the function ≤ can take on an unlimited number of values, but it is only necessary to consider three cases: if the arguments are arranged in ascending order according to value, then the order is either $X$ $Y$, in which case the result of the function $X{\le}Y$ is 1, or the order is $Y$ $X$ in which case the result of $X{\le}Y$ is 0, or the two are equal, in which case the result is 1. This may be summarized in a table as follows:

| Case | $X{\le}Y$ |
|---|---|
| $X$ $Y$ | 1 |
| $Y$ $X$ | 0 |
| $Y{=}X$ | 1 |

For example, if $X{\leftarrow}5$ and $Y{\leftarrow}3$ then the values $X$ and $Y$ arranged in ascending sequence appear in the order $Y$ $X$. The appropriate case is therefore the second row of the body of the table and the result of the function $X{\le}Y$ may be read off as 0. Similarly, if $X{\leftarrow}5$ and $Y{\leftarrow}5$, then the appropriate case is $Y{=}X$ and the result appears in the last row of the table.

Moreover, if a column for the expression $Y\le X$ is added, the table appears as shown in Table 14.3. This table shows that the function $\le$ is not commutative.

| Case | $X\le Y$ | $Y\le X$ |
|---|---|---|
| $X\ Y$ | 1 | 0 |
| $Y\ X$ | 0 | 1 |
| $Y=X$ | 1 | 1 |

Non-Commutativity of $\le$

Table 14.3

The same scheme of exhaustion can be used to determine the commutativity of the other relations $<$ $=$ $\ge$ $>$ and $\ne$, and of the functions $\lceil$ and $\lfloor$. For example, Table 14.4 shows that maximum is commutative.

| Case | $X\lceil Y$ | $Y\lceil X$ |
|---|---|---|
| $X\ Y$ | $Y$ | $Y$ |
| $Y\ X$ | $X$ | $X$ |
| $Y=X$ | $X$ | $X$ |

Commutativity of $\lceil$

Table 14.4 ⊡4

## 14.3 ASSOCIATIVITY

Since $X+(Y+Z)$ yields the same result as $(X+Y)+Z$, the function $+$ is said to be associative. Multiplication is also associative, that is,

$$(X\times(Y\times Z))=((X\times Y)\times Z)$$

It is easy to show that subtraction and division are not associative. For example, $4-(3-2)$ yields $3$ and $(4-3)-2$ yields $^{-}1$.

The associativity of the maximum function can be established by examining all possible cases. If three names $X$, $Y$, and $Z$ are arranged in non-decreasing order according to their values, they can occur in exactly six possible arrangements. These are shown in Table 14.5, together with columns showing the evaluation of the expression $X\lceil(Y\lceil Z)$ and $(X\lceil Y)\lceil Z$. This evaluation proceeds as follows. The first column shows the values of the expression $X\lceil Y$, and the second shows the maximum of these values and $Z$; the third column shows the values of $Y\lceil Z$, and the fourth column shows the maximum of $X$ and these values. Since columns 2 and 4 agree, the function $\lceil$ is associative.

| Case | | | X Y | (X⌈Y)⌈Z | Y⌈Z | X⌈(Y⌈Z) |
|---|---|---|---|---|---|---|
| X | Y | Z | Y | Z | Z | Z |
| X | Z | Y | Y | Y | Y | Y |
| Y | X | Z | X | Z | Z | Z |
| Y | Z | X | X | X | Z | X |
| Z | X | Y | Y | Y | Y | Y |
| Z | Y | X | X | X | Y | X |

Associativity of ⌈

5-7⌹ Table 14.5

## 14.4 DISTRIBUTIVITY

The identity

```
(X×(Y+Z))=((X×Y)+(X×Z))
```

is said to represent the distributivity of multiplication over addition, since it shows that the effect of multiplication by *X* on the sum *Y+Z* (shown to the left of the equal sign) can be said to distribute equally over each of the arguments *Y* and *Z* as shown on the right.

To see why multiplication distributes over addition, it is helpful to use the picture of multiplication presented in the discussion of commutativity, that is, the product of two factors *P* and *Q* is pictured as the number of elements in the array (*P*,*Q*) ρ '□'. The left side of the identity of the preceding paragraph is then represented by the array (*X*,*Y*+*Z*)ρ'□', and the right side by the sum of the arrays (*X*,*Y*)ρ'□' and (*X*,*Z*)ρ'□'. For example, if *X*←4 and *Y*←9 and *Z*←5, then:

```
 (X,Y+Z)ρ'□'
□□□□□□□□□□□□□□
□□□□□□□□□□□□□□
□□□□□□□□□□□□□□
□□□□□□□□□□□□□□

(X,Y)ρ'□'  (X,Z)ρ'□'
□□□□□□□□□  □□□□□
□□□□□□□□□  □□□□□
□□□□□□□□□  □□□□□
□□□□□□□□□  □□□□□
```

If the last two arrays are pushed together they form an array identical to the first and therefore contain the same
8-9⌹ total number of elements as the first.

The function and distributes over or, that is:

```
(X∧(Y∨Z))=((X∧Y)∨(X∧Z))
```

Since the arguments $X$, $Y$, and $Z$ are each limited to the values 0 and 1, this identity can be examined by evaluating the expressions for each of the eight possible cases as shown in Table 14.6.

| $X$ | $Y$ | $Z$ | $Y\vee Z$ | $X\wedge(Y\vee Z)$ | $X\wedge Y$ | $X\wedge Z$ | $(X\wedge Y)\vee(X\wedge Z)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Distributivity of $\wedge$ over $\vee$

Table 14.6 ⌹10-12

The function $\lceil$ distributes over $\lfloor$, that is,

$(X\lceil(Y\lfloor Z))=((X\lceil Y)\lfloor(X\lceil Z))$

To examine this putative identity, it is necessary to consider the six possible arrangements of the arguments $X$, $Y$, and $Z$ when arranged in non-decresing order according to value. This is shown in Table 14.7.

| Case | | | $Y\lfloor Z$ | $X\lceil(Y\lfloor Z)$ | $X\lceil Y$ | $X\lceil Z$ | $(X\lceil Y)\lfloor(X\lceil Z)$ |
|---|---|---|---|---|---|---|---|
| $X$ | $Y$ | $Z$ | $Y$ | $Y$ | $Y$ | $Z$ | $Y$ |
| $X$ | $Z$ | $Y$ | $Z$ | $Z$ | $Y$ | $Z$ | $Z$ |
| $Y$ | $X$ | $Z$ | $Y$ | $X$ | $X$ | $Z$ | $X$ |
| $Y$ | $Z$ | $X$ | $Y$ | $X$ | $X$ | $X$ | $X$ |
| $Z$ | $X$ | $Y$ | $Z$ | $X$ | $Y$ | $X$ | $X$ |
| $Z$ | $Y$ | $X$ | $Z$ | $X$ | $X$ | $X$ | $X$ |

Distributivity of $\lceil$ over $\lfloor$

Table 14.7

A function may distribute over itself. For example, the function $\lfloor$ does so:

$(X\lfloor(Y\lfloor Z))=((X\lfloor Y)\lfloor(X\lfloor Z))$

This fact can be examined by means of a table similar to Table 14.7. It can easily be shown that plus does not distribute over itself. For example, $3+(4+5)$ is not equal to $(3+4)+(3+5)$.

The distributivity properties of functions can be summarized conveniently in a table. For example, for the functions + × ⌈ and ⌊, the results derived thus far are shown in Table 14.8. For example, the second row (labelled ×), shows that × distributes over +. The blank entries of the table could be filled in by further analysis. For example, plus does not distribute over either itself or times, but it does distribute over both maximum and minimum; the complete first row of Table 14.8 would therefore be 0 0 1 1.

| | + | × | ⌈ | ⌊ |
|---|---|---|---|---|
| + | 0 | | | |
| × | 1 | | | |
| ⌈ | | | | 1 |
| ⌊ | | | | 1 |

Some distributivity properties

13-15⊡ Table 14.8

## 14.5 IDENTITIES BASED ON COMMUTATIVITY, ASSOCIATIVITY, AND DISTRIBUTIVITY

It is important to recognize that an identity such as $(X\times Y)=(Y\times X)$ applies not only to the simple names $X$ and $Y$, but also to any expression that may be substituted for them. For example, if the expression $(P\times Q-R)$ is substituted for $X$, and the expression $(M+R\times Q)$ is substituted for $Y$, then the foregoing identity (representing the commutativity of multiplication) ensures that

$(P\times Q-R)\times(M+R\times Q)$

is equivalent to

$(M+R\times Q)\times(P\times Q-R)$

The combined use of the properties of commutativity, associativity and distributivity leads to a host of identities too numerous to list. For example, $(A+B)\times C$ is equivalent to $C\times(A+B)$ (since × is commutative), which is equivalent to $(C\times A)+(C\times B)$ (since × distributes over +), which is equivalent to $(A\times C)+(B\times C)$ (since × is commutative). Consequently, $(A+B)\times C$ is equivalent to $(A\times C)+(B\times C)$.

In order to show the proof of such a result clearly, it is convenient to simply list the successive equivalent statements, one below the other, together with notes to the right of them showing what property was used to justify each new equivalent statement. For example, the proof used in the preceding paragraph would be shown as follows:

$$\begin{array}{ll}
(A+B)\times C & \\
C\times(A+B) & \text{Commutativity of } \times \\
(C\times A)+(C\times B) & \text{Distributivity of } \times \text{ over } + \\
(A\times C)+(B\times C) & \text{Commutativity of } \times
\end{array}$$

A proof can be <u>illuminated</u> by evaluating each of the expressions occurring in it for some chosen values of their arguments. For example, if $A\leftarrow 3$ and $B\leftarrow 5$ and $C\leftarrow 7$, the illumination of the foregoing proof would appear as follows:

```
      (3+5)×7
56
      7×(3+5)
56
      (7×3)+(7×5)
56
      (3×7)+(5×7)
56
```
⌹16-13

For convenience, the notes written to justify each step in a proof will be abbreviated; the symbols $\underline{C}$, $\underline{A}$, and $\underline{D}$ will be used to denote commutativity, associativity and distributivity. Thus $\underline{C}+$ means that $+$ is commutative, $\underline{A}\times$ means that $\times$ is associative, and $\times\underline{D}+$ means that $\times$ distributes over $+$.

The following shows the use of these abbreviations in the proof of a rather important identity:

$$\begin{array}{ll}
(A+B)\times(C+D) & \\
((A+B)\times C)+((A+B)\times D) & \times\underline{D}+ \\
(C\times(A+B))+(D\times(A+B)) & \underline{C}\times \\
((C\times A)+(C\times B))+((D\times A)+(D\times B)) & \times\underline{D}+ \\
((A\times C)+(B\times C))+((A\times D)+(B\times D)) & \underline{C}\times \\
(A\times C)+((B\times C)+(A\times D))+(B\times D) & \underline{A}+ \\
(A\times C)+((A\times D)+(B\times C))+(B\times D) & \underline{C}+ \\
(A\times C)+(A\times D)+(B\times C)+(B\times D) & \underline{A}+
\end{array}$$

Consequently, the first expression, $(A+B)\times(C+D)$, is equivalent to the last, $(A\times C)+(A\times D)+(B\times C)+(B\times D)$, that is:

$$\begin{array}{l}
(A+B)\times(C+D) \\
(A\times C)+(A\times D)+(B\times C)+(B\times D)
\end{array}$$

In other words, each element of the first sum is multiplied by each element of the second sum and the four resulting terms are added together. ⌹19-21

The foregoing result will be used in proving further results, and to make it easy to refer to, it will be given the name <u>Theorem 1</u>. One reason for the importance of Theorem 1 is that it has some useful special cases. For example, if $A$ and $C$ both have the same value $X$, then according to Theorem 1, the expression (X+B)×(X+D) is equivalent to (X×X)+(X×D)+(B×X)+(B×D). This leads to the following derivation:

| | |
|---|---|
| (X+B)×(X+D) | |
| (B+X)×(D+X) | C+ |
| (B×D)+(B×X)+(X×D)+(X×X) | Theorem 1 |
| (B×D)+((B×X)+(X×D))+(X×X) | A+ |
| (B×D)+((X×B)+(X×D))+(X×X) | C× |
| (B×D)+(X×(B+D))+(X×X) | ×D+ |
| (B×D)+((B+D)×X)+(X×X) | C× |
| (B×D)+((B+D)×X)+(X*2) | (X*2)=(X×X) |
| (B×D)+((B+D)×(X*1))+(X*2) | (X*1)=X |
| ((B×D)×X*0)+((B+D)×X*1)+(X*2) | (X*0)=1 |
| +/((B×D)×X*0),((B+D)×X*1),(X*2) | (P+Q+R)=+/P,Q,R |
| +/((B×D),(B+D),1)×X*0 1 2 | ((P[1]×Q[1])+(P[2]×Q[2]) +(P[3]×Q[3]))=+/P×Q |

Finally then:

```
(X+B)×(X+D)
+/((B×D),(B+D),1)×X*0 1 2
```

In other words, (X+B)×(X+D) is equivalent to a polynomial in X with the coefficients B×D and B+D and 1.

For example, if B is 2 and D is 3, the polynomial has the coefficients 6, 5, and 1. In other words:

```
((X+2)×(X+3))=(+/6 5 1×X*0 1 2)
```

The product (X+2)×(X+3) can also be expressed in the form ×/X+2 3. In general if V is any two-element vector, then ×/X+V is equivalent to (X+V[1])×(X+V[2]). Moreover, the coefficients of the equivalent polynomial are given by ×/V and +/V and 1. That is:

22-23⊡ (×/X+V)=+/((×/V),(+/V),1)×X*0 1 2

## 14.6 IDENTITIES ON VECTORS

Thus far, the identities considered have been applied only to scalar arguments. However, many of them apply equally to vectors. For example, the commutativity of × assures that (A×B)=(B×A) and that 3×5 is therefore equal to 5×3. However, if A is the vector 3 5 7 and B is the vector 2 0 ¯1, it is still true that (A×B)=(B×A). For example:

```
      A←3 5 7
      B←5 0 ¯1
      A×B
15 0 ¯7
      B×A
15 0 ¯7
```

Commutativity of × applies for vectors because it applies for each of the corresponding pairs of elements of the arguments.

For the same reason, the associativity and distributivity of functions apply to vectors as well. For example:

```
      A←3  5  7
      B←5  0  ¯1
      C←6  4  2
      (A⌈B)⌈C
6  5  7
      A⌈(B⌈C)
6  5  7
      A×(B+C)
33  20  7
      (A×B)+(A×C)
33  20  7
      A⌊B
3  0  ¯1
      C+(A⌊B)
9  4  1
      C+A
9  9  9
      C+B
11  4  1
      (C+A)⌊(C+B)
9  4  1
```

⌹24-25

There are also some important identities concerning the reduction of vectors. Thus (+/*A*)+(+/*B*) is equivalent to +/*A*,*B*. For example:

```
      (+/1 2 3)+(+/4 5 6 7)
      (1+2+3)+(4+5+6+7)          Definition of +/
      1+2+3+4+5+6+7              A+
      +/1 2 3 4 5 6 7            Definition of +/
      +/(1 2 3),(4 5 6 7)        Definition of ,
```

Moreover, if the vectors *A* and *B* are of the same dimension so that *A*+*B* is meaningful, then (+/*A*)+(+/*B*) is equivalent to +/*A*+*B*. For example, if *A* is 1 2 3 and *B* is 4 5 6:

```
  (+/1 2 3)+(+/4 5 6)
  (1+2+3)+(4+5+6)            Definition of +/
  1+2+(3+4)+5+6              A+
  1+2+(4+3)+5+6              C+
  1+(2+4)+(3+5)+6            A+
  1+(4+2)+(5+3)+6            C+
  (1+4)+(2+5)+(3+6)          A+
  +/(1+4),(2+5),(3+6)        Definition of +/
  +/1 2 3+4 5 6              Definition of vector addition
```

Since the only properties of addition used in the foregoing derivations were its commutativity and associativity, the same results hold for any function which is both commutative and associative. For example:

```
((⌈/A)⌈(⌈/B))=(⌈/A,B)
((⌈/A)⌈(⌈/B))=(⌈/A+B)
((×/A)×(×/B))=(×/A,B)
((×/A)×(×/B))=(×/A×B)
```

Thus if *F* is any function which is both associative and commutative, then

```
((F/A)F(F/B))=(F/A,B)
```

Since this is a very useful result which will be referred to again in later derivations, it will be given the name Theorem 2.

Moreover, if *F* is any function which is both associative and commutative, and *A* and *B* are vectors of the same dimension, then

```
((F/A)F(F/B))=(F/A F B)(Theorem 3)
```

This result will be called Theorem 3, as indicated by the
26-27⌹ note to the right of the identity.

Since × distributes over +, a product of sums can be expressed as a sum of products. More explicitly, if *V* and *W* are two vectors, then

```
((+/V)×(+/W))=+/+/V∘.×W      (Theorem 4)
```

For example:

```
      V←3  1  4
      W←5  0  2  6
      (+/V)×(+/W)
104
      V∘.×W
15  0  6  18
 5  0  2   6
20  0  8  24

      +/V∘.×W
39 13 52
      +/+/V∘.×W
```
28⌹ 104

The preceding identity (Theorem 4) and the following one will both be useful in the treatment of products of polynomials:

```
((A×P)∘.×(B×Q))=((A∘.×B)×(P∘.×Q))      (Theorem 5)
```

For example:

```
       A←1  2  3
       B←4  5  6  7
       P←2  0  2
       Q←3  1  3  1
       A×P
2  0  6
       B×Q
12  5  18  7
       (A×P)∘.×(B×Q)
24   10   36   14
 0    0    0    0
72   30  108   42

       A∘.×B
 4   5   6   7
 8  10  12  14
12  15  18  21

       P∘.×Q
6  2  6  2
0  0  0  0
6  2  6  2

       (A∘.×B)×(P∘.×Q)
24   10   36   14
 0    0    0    0
72   30  108   42
```

Each side of the identity of Theorem 5 is a table; the identity will be derived by showing that (for any value of *I* and any value of *J*) the element in the *I*th row and *J*th column of the table on the left is identical with the corresponding element of the table on the right:

```
 ((A×P)∘.×(B×Q))[I;J]
 ((A×P)[I])×((B×Q)[J])            Definition of ∘.×
 (A[I]×P[I])×(B[J]×Q[J])          Multiplication of vectors
 A[I]×(P[I]×B[J])×Q[J]            A×
 A[I]×(B[J]×P[I])×Q[J]            C×
 (A[I]×B[J])×(P[I]×Q[J])          A×
 ((A∘.×B)[I;J])×((P∘.×Q)[I;J])    Definition of ∘.×
 ((A∘.×B)×(P∘.×Q))[I;J]           Multiplication of tables
```

The only properties of the function × used in this derivation are its associativity and commutativity. Therefore, the same derivation would apply for any function which is both assocative and commutative. Hence Theorem 5 remains true if any such function is substituted for ×. For example:

```
       ((A⌈P)∘.⌈(B⌈Q))=((A∘.⌈B)⌈(P∘.⌈Q))
```
⍠29-31

## 14.7 THE POWER FUNCTION

Consider the following expressions:

```
        2*3
8
        2*4
16
        (2*3)×(2*4)
128
        2*(3+4)
128
        (2*(3+4))=((2*3)×(2*4))
1
```

The foregoing result suggests the following identity:

```
        (A*(B+C))=((A*B)×(A*C))          (Theorem 6)
```

It can be derived as follows:

```
        (A*B)×(A*C)
        (×/BρA)×(×/CρA)          (P*Q)=×/QρP
        ×/(BρA),×(CρA)           Theorem 2
        ×/(B+C)ρA                Definitions of ρ and ,
32⍠     A*(B+C)                  (P*Q)=×/QρP
```

Theorem 6 leads to a very useful identity on vectors. If $X$ is a scalar and $E$ and $F$ are any vectors, then:

```
        ((X*E)∘.×(X*F))=(X*E∘.+F)   (Theorem 7)
```

For example:

```
        E←0  1  2
        F←0  1  2  3
        X←2
        X*E
1  2  4
        X*F
1  2  4  8
        (X*E)∘.×(X*F)
1   2   4   8
2   4   8  16
4   8  16  32
        E∘.+F
0  1  2  3
1  2  3  4
2  3  4  5
        X*E∘.+F
1   2   4   8
2   4   8  16
33⍠ 4   8  16  32
```

# 15

# Identities on Polynomials

## 15.1 INTRODUCTION

In the introduction to polynomials in Section 13.5 it was remarked that the importance of polynomials rests not only on the facts that they can be evaluated easily and can be used to approximate any function of practical interest, but also on the fact that they are easily analyzed: the sum of two polynomials is equivalent to a polynomial, the product of two polynomials is equivalent to a polynomial, a factorial polynomial is equivalent to a polynomial, and an expression of the form (*X*+1)⋆*N* is equivalent to a polynomial. Each of these equivalences is derived in the present chapter. The theorems cited are those of Chapter 14.

The polynomial was defined in Section 13.5 by the expression +/*C*×*X*⋆¯1+ιρ,*C* where *C* is the vector of coefficients and *X* is the argument. This expression applies only to a scalar argument *X*, and a more general expression applying to a vector argument *X* was derived in Section 13.6 and defined as the function *P* as follows:

```
∇Z←C P X
Z←(X∘.⋆¯1+ιρ,C)+.×C∇
```

## 15.2 THE SUM OF POLYNOMIALS

Consider the polynomials 1 3 5 *P X* and 6 1 4 *P X*. Their sum can be shown to be equivalent to the polynomial 7 4 9 *P X* whose coefficient vector is the sum of the coefficient vectors of the given polynomials, that is:

```
((1 3 5 P X)+(6 1 4 P X))=((1 3 5+6 1 4)P X)
```

In general, if *X* is a scalar and *A*, *B* and *E* are vectors of the same dimension, then

```
((+/A×X⋆E)+(+/B×X⋆E))=(+/(A+B)×X⋆E)
```

In particular, if *E* is the vector ¯1+ιρ*A*, then the left side of the foregoing identity is the sum of the polynomial with coefficients *A* and the polynomial with coefficients *B* , and the right side is the polynomial with coefficients A+B. The derivation of the identity follows:

```
(+/A×X*E)+(+/B×X*E)
+/(A×X*E)+(B×X*E)        Theorem 3
+/((X*E)×A)+((X*E)×B)    C×
+/(X*E)×(A+B)            ×D+
+/(A+B)×(X*E)            C×
```

The polynomials C P X and (C,0) P X are clearly equivalent, since an extra term in the polynomial with a zero coefficient will contribute nothing to the sum. For example, if C←1 2 3, and X←4, then:

```
C P X
+/1 2 3×4*0 1 2
+/1 2 3×1 4 16
+/1 8 48
57
```

and

```
(C,0) P X
+/1 2 3 0×1 4 16 64
+/1 8 48 0
57
```

More generally, any number of zeros may be appended to the right of a vector of coefficients without changing the polynomial, that is, ((C,Nρ0) P X)=(C P X). Consequently, two polynomials with coefficients C and D of different dimensions may be added by first appending enough zeros to the shorter of the two to yield a vector of the same dimension as the longer. For example, if (ρD)<ρC, then:

```
((C+(ρC)↑D) P X)=(C P X)+(D P X)
```

The following identity applies to every case, that is, for (ρD) less than, equal to, or greater than ρC:

```
      M←(ρC)⌈(ρD)
1-2   (((M↑C)+(M↑D)) P X)=(C P X)+(D P X)
```

## 15.3 THE PRODUCT OF POLYNOMIALS

The product of two polynomials is equivalent to another polynomial whose coefficients are easily determined from the coefficients of the given polynomials. In other words,

```
(E P X)=((C P X)×(D P X))
```

and the coefficients E can be determined from C and D. The method will first be described by means of an example and the derivation will be shown later.

Suppose that C←3 1 4 and D←2 0 5 3. First form the multiplication table C∘.×D whose value is:

```
6  0 15  9
2  0  5  3
8  0 20 12
```

Then draw diagonal lines through the table and sum the numbers on each diagonal, placing each sum at the end of its diagonal as shown below:

```
              C∘.×D
              6  0 15  9
              2  0  5  3
              8  0 20 12

6  2 23 14 23 12
```

The result is the vector of coefficients 6 2 23 14 23 12; that is: (6 2 23 14 23 12 P X)=(3 1 4 P X)×(2 0 5 3 P X) ⌹3-4

We begin with an informal proof which will expose the basic notion employed in the succeeding formal proof. Consider the multiplication table C∘.×D bordered on the left by C and above by D, and bordered to the right by the exponents of X associated with each of the coefficients C, and bordered below by the exponents associated with the coefficients D:

```
      2   0   5   3
 3  | 6   0  15   9 |  0
 1  | 2   0   5   3 |  1
 4  | 8   0  20  12 |  2
      0   1   2   3
```

The product of the polynomials C P X and D P X is a sum of the products formed from each term of the first polynomial with each term of the second. These products are therefore of the form:

```
(C[I]×(X*I-1)×(D[J]×X*J-1)
(C[I]×D[J])×(X*I-1)×(X*J-1)          C×
C[I]×D[J]×X*((I-1)+(J-1))            Theorem 6
```

The product C[I]×D[J] is the entry in row I and column J of the table, and the exponent associated with it is therefore (I-1)+(J-1), that is, the sum of the corresponding bordering elements on the right and below. Hence the exponents associated with the body of the table are those in the table 0 1 2∘.+0 1 2 3 whose value is:

```
0  1  2  3
1  2  3  4
2  3  4  5
3  4  5  6
```

From this it is clear that the products corresponding to any one value of the exponent lie along a diagonal and are therefore summed as shown at the beginning of this section.

A formal proof will now be presented. The product of the polynomials `C P X` and `D P X` may be written as:

```
(+/C×X*¯1+⍳⍴C)×(+/D×X*¯1+⍳⍴D)
```

In this form it is clear that the product is a product of the sums of two vectors `V` and `W`, where `V←C×X*¯1+⍳⍴C` and `W←D×X*¯1+⍳⍴D`, that is, `(+/V)×(+/W)`. The results of Theorem 4 can therefore be applied to express the result in terms of the multiplication table for `V` and `W`:

```
((+/V)×(+/W))=+/+/V∘.×W
```

Since `V` is the product of two vectors (that is, `C` and `X*¯1+⍳⍴C`) and `W` is the product of two vectors, Theorem 5 can be applied to write the table `V∘.×W` as the product of the two tables `C∘.×D` and `(X*¯1+⍳⍴C)∘.×(X*¯1+⍳⍴D)`. That is:

```
(V∘.×W)=(C∘.×D)×((X*¯1+⍳⍴C)∘.×(X*¯1+⍳⍴D))
```

But Theorem 7 allows us to write `X*(¯1+⍳⍴C)∘.+(¯1+⍳⍴D)` for the second table; that is,

```
(V∘.×W)=(C∘.×D)×X*(¯1+⍳⍴C)∘.+(¯1+⍳⍴D)
```

For example, if `C` and `D` are as defined in the earlier example (that is, `C←3 1 4` and `D←2 0 5 3`), then:

```
      C∘.×D                      (¯1+⍳⍴C)∘.+(¯1+⍳⍴D)
6   0  15   9                    0   1   2   3
2   0   5   3                    1   2   3   4
8   0  20  12                    2   3   4   5
```

The table on the right gives the exponents of `X`.

To summarize:

```
(C P X)×(D P X)
(+/C×X*¯1+⍳⍴C)×(+/D×X*¯1+⍳⍴D)        Definition of polynomial
+/+/(C×X*¯1+⍳⍴C)∘.×(D×X*¯1+⍳⍴D)      Theorem 4
+/+/(C∘.×D)×(X*¯1+⍳⍴C)∘.×(X*¯1+⍳⍴D)  Theorem 5
+/+/(C∘.×D)×X*(¯1+⍳⍴C)∘.+(¯1+⍳⍴D)    Theorem 7
```

It is clear that the table of exponents `(¯1+⍳⍴C)∘.+(¯1+⍳⍴D)` will always be of the form shown in the example in the preceding paragraph, that is, it contains a zero in the upper left corner, `1`'s in the next diagonal, `2`'s in the next diagonal, and so on. Hence the element of the table `C∘.×D` that is multiplied by `X*0` is in the upper left hand corner, the elements multiplied by `X*1` are on the next diagonal, etc. Hence the appropriate coefficients for `X*0` and X⋆1, and X⋆2, etc., in the product polynomial are obtained as the upper left corner of C∘.×D, the sum of the next diagonal of C∘.×D, the sum of the next diagonal, etc. This is the pattern shown in the rule given at the outset for multiplying polynomials. ⌹5

## 15.4 THE PRODUCT ×/X+V

In Section 14.5 it was shown that the product (X+2)×(X+3) could be expressed in the form ×/X+2 3, and that, more generally, if V were any 2-element vector, then ×/X+V was equivalent to (X+V[1])×(X+V[2]). Moreover, it was shown that ×/X+V was equivalent to the polynomial with coefficients (×/V),(+/V),1. The case of a vector V of arbitrary dimension will now be considered.

The expression X+2 is equivalent to the polynomial with coefficients 2 1, that is, (X+2)=+/2 1×X⋆0 1. Similarly, X+3 is equivalent to the polynomial with coefficients 3 1. Therefore, the product (X+2)×(X+3) can be treated as a product of polynomials. The coefficients of the product polynomial may then be obtained by the method of Section 15.3 as follows:

```
      2 1∘.×3 1

           6   2
           3   1
 6    5    1
```

This result agrees with that obtained in Section 14.5.

Consider now the product ×/X+4 2 3:

```
      ×/X+4 2 3
      (X+4)×(X+2)×(X+3)        Definition of ×/
      (X+4)×(6 5 1 P X)        Preceding result
      (4 1 P X)×(6 5 1 P X)    X+4 as a polynomial
```

This last product of polynomials can again be evaluated by the method of the earlier section:

```
                4 1∘.×6 5 1

               24  20   4
                6   5   1
 24   26   9    1
```

Hence (×/X+4 2 3)=(24 26 9 1) P X ⌹6

It should now be clear that the product ×/X+V is a product of polynomials with coefficients V[1],1 and V[2],1 and V[3],1, etc. The coefficients of a polynomial equivalent to ×/X+V can therefore be obtained by multiplying these polynomials together in turn. The following function QA produces the desired coefficients as a function of the vector V:

```
      ∇Z←QA V
[1]   Z←1
[2]   V←,V
[3]   I←ρV
[4]   Z←(V[I]×Z,0)+(0,Z)
[5]   I←I-1
[6]   →3×I≠0∇
```

For example:

```
      TΔQA←3
      Q 4 2 3
QA[3] 3 1
QA[3] 6 5 1
QA[3] 24 26 9 1
24 26 9 1
```

Finally, then:

```
7-10⍠      (×/X+V)=(QA V)P X
```

## 15.5 BINOMIAL COEFFICIENTS

The function (X+1)*4 is equivalent to the function

```
      (X+1)×(X+1)×(X+1)×(X+1)
```

and is therefore equivalent to the function

```
      ×/X+1 1 1 1
```

or

```
      ×/X+4ρ1
```

More generally:

```
      ((X+1)⋆N)=×/X+Nρ1
```

The last result of Section 15.4 stated that

```
      (×/X+V)=(QA V)P X
```

For the case V←Nρ1 this becomes

```
      (×/X+Nρ1)=(QA Nρ1)P X
```

Combining this result with the final result of the preceding paragraph yields the following important identity:

```
((X+1)*N)=(QA Nρ1)P X
```
⌹11

The results of Exercise 15.8 suggest that the function *QA* applied to the argument *N*ρ1 yields the same result as the simpler function *BIN* of Section 12.4 when applied to the argument *N*. In other words, they suggest that:

```
∧/(QA Nρ1)=BIN N
```

The reason for this can be seen by examining the two functions repeated below:

```
      ∇Z←QA V
[1]   Z←1
[2]   V←,V
[3]   I←ρV
[4]   Z←(V[I]×Z,0)+(0,Z)
[5]   I←I-1
[6]   →3×I≠0∇
```

```
      ∇Z←BIN X
[1]   Z←,1
[2]   →3×X≥ρZ
[3]   Z←(Z,0)+(0,Z)
[4]   →2∇
```

If each element of the argument *V* is equal to 1, then line 4 of the function *QA* is equivalent to line 3 of the function *BIN*. Moreover, this line of each function is repeated the same number of times because ρ*V* is equal to *N* if *V*←*N*ρ1. ⌹12

## 15.6 THE FACTORIAL POLYNOMIALS

The factorial polynomials introduced in Section 10.8 for the purpose of fitting functions were defined as follows:

| Degree of Factorial Polynomial | Factorial Polynomial |
|---|---|
| 0 | 1 |
| 1 | X |
| 2 | X×(X-1) |
| 3 | X×(X-1)×(X-2) |
| 4 | X×(X-1)×(X-2)×(X-3) |

Such a polynomial can also be written in the form ×/*X*+*V*, where *V* is the vector 1-ι*N* and *N* is the degree of the polynomial.

The coefficients of a polynomial equivalent to the factorial polynomial of degree *N* can therefore be obtained by applying the function *QA* to the argument 1-ι*N*. For example:

```
      QA -0
0  1
      QA -0  1
0  ¯1  1
      QA -0  1  2
0  2  ¯3  1
      QA -0  1  2  3
0  ¯6  11  ¯6  1
```

Hence:

```
            (0 1 P X)=X
         (0 ¯1 1 P X)=X×(X-1)
       (0 2 ¯3 1 P X)=X×(X-1)×(X-2)
   (0 ¯6 11 ¯6 1 P X)=X×(X-1)×(X-2)×(X-3)
```

In the introduction to Chapter 14 it was shown that the function +/(ι*X*)*2 (that is, the sum of the squares of the integers to *X*) was equivalent to the following sum of factorial polynomials:

```
   0+X+((3÷2)×X×(X-1))+(2÷6)×X×(X-1)×(X-2)
```

Moreover, it was stated that this expression was equivalent to the polynomial (÷6)×(*X**0 1 2 3)+.×0 1 3 2. This statement can now be proven as follows:

```
0+X+((3÷2)×X×(X-1))+(2÷6)×X×(X-1)×(X-2)
(÷6)×6×(X+((3÷2)×X×(X-1))+(2÷6)×X×(X-1)×(X-2))          1=(÷6)×6
(÷6)×((6×X)+(9×X×(X-1))+(2×X×(X-1)×(X-2)))              ×D+
(÷6)×((6×0 1 P X)+(9×0 ¯1 1 P X)+(2×0 2 ¯3 1 P X))      Note 1
(÷6)×((0 6 P X)+(0 ¯9 9 P X)+(0 4 ¯6 2 P X))            Note 2
(÷6)×((0 6 0 0 P X)+(0 ¯9 9 0 P X)+(0 4 ¯6 2 P X))      Note 3
(÷6)×(0 1 3 2 P X)                                      Note 4
(÷6)×+/0 1 3 2×X*0 1 2 3                                Note 5
(÷6)×+/(X*0 1 2 3)×0 1 3 2                              C×
(÷6)×(X*0 1 2 3)+.×0 1 3 2                              Note 6
```

Note 1: Polynomial equivalent of factorial polynomials
Note 2: (*A*×(*C P X*))=(*A*×*C*) *P X*
Note 3: ((*C*,0) *P X*)=*C P X*
Note 4: Sum of polynomials
Note 5: Definition of Polynomials
13-14⊟ Note 6: Definition of +.×

## 15.7 MATHEMATICAL INDUCTION

The function +/ι*X* can be analyzed by constructing a difference table as follows:

| $X$ | $+/\iota X$ | $D+/\iota X$ | $D\ D+/\iota X$ | $D\ D\ D+/\iota X$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 2 | 1 | 0 |
| 2 | 3 | 3 | 1 | |
| 3 | 6 | 4 | | |
| 4 | 10 | | | |

The results of Section 10.8 may then be applied to conclude that the function $+/\iota X$ was equivalent to the following sum of factorial polynomials:

$$0+X+(.5\times X\times(X-1))$$

In drawing this conclusion it is assumed that every one of the third differences (in the last column) would be 0. This happens to be true for the function $+/\iota X$, but the calculations of this table do not prove it to be so.

For example, suppose one attempted to analyze the function

$$Y\leftarrow X+(.5\times X\times X-1)+X\times(X-1)\times(X-2)\times(X-3)\times(X-4)$$

The first five entries in the difference table would appear exactly the same as the table shown for $+/\iota X$, and one might erroneously conclude that all third differences would be zero. However, if one considered one further row, the table would appear as follows:

| $X$ | $Y$ | $D\ Y$ | $D\ D\ Y$ | $D\ D\ D\ Y$ | $D\ D\ D\ D\ Y$ | $D\ D\ D\ D\ D\ Y$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 120 |
| 1 | 1 | 2 | 1 | 0 | 120 | |
| 2 | 3 | 3 | 1 | 120 | | |
| 3 | 6 | 4 | 121 | | | |
| 4 | 10 | 125 | | | | |
| 5 | 135 | | | | | |

A difference table can yield the coefficients of a polynomial which fits a given function exactly for a certain number of values of the argument and which probably fits it very nearly or exactly for all values of the argument, but study of the difference table alone cannot ensure that it fits for all points. It is therefore desirable to develop other means of verifying that an expression derived from a difference table does in fact agree with the given function for points other than those actually used in the table.

Let us suppose that the functions $+/\iota X$ and $X+.5\times X\times X-1$ do agree for some integer value $K$, that is, we suppose that

$$(+/\iota K)=K+.5+K\times K-1$$

From this assumption alone, we will now show that they must agree for the argument $K+1$.

We have undertaken to show that `+/⍳K+1` is equal to `(K+1)+.5×(K+1)×(K+1)-1`, in other words to show that

```
(+/⍳K+1)-((K+1)+.5×(K+1)×(K+1)-1)
```

is zero.

Let the functions *F* and *G* be defined as follows:

```
∇Z←F X                    ∇Z←G X
Z←+/⍳X∇                   Z←X+.5×X×X-1∇
```

We wish to show that *F* and *G* agree for all integer values of their argument, that is, that `(F X)-(G X)` is zero for every integer *X*. We begin by expressing the difference for the argument `K+1` in terms of the difference for argument *K* as follows:

```
(F K+1)-(G K+1)
(+/⍳K+1)-((K+1)+.5×(K+1)×(K+1)-1)   Definitions of F and G
((+/⍳K)+(K+1))-((K+1)+.5×(K+1)×K)   (+/⍳K+1)=(+/⍳K)+K+1
((+/⍳K)+(K+1)-(K+1))-.5×(K+1)×K
((+/⍳K)+0)-.5×(K+1)×K
(+/⍳K)-.5×(2×K)+(K-1)×K
(+/⍳K)-K+.5×K×K-1
(F K)-(G K)                          Definitions of F and G
```

Hence the difference between `F K+1` and `G K+1` must be the same as the difference between `F K` and `G K`. In other words, if `F K` and `G K` are equal, then `F K+1` and `G K+1` must also be equal.

But for `K=1`, `F K` and `G K` are obviously equal; that is `+/⍳1` is equal to `1+.5×1×0`. Hence `F 1+1` must equal `G 1+1`, that is, `F 2` equals `G 2`. Thus, for `K=2`, `F K` equals `G K`. Therefore `F 2+1` equals `G 2+1`, and so on for all possible integer arguments. Hence `F X` equals `G X` for all positive integer values of *X*.

This method of proof is called <u>mathematical induction</u>. To prove that two function *F* and *G* are equivalent, proceed as follows:

1) Show that the difference `(F K+1)-(G K+1)` is equal to the difference `(F K)-G K`.

2) Show that `F 1` is equal to `G 1`.

If items 1 and 2 can both be shown to be true then the
15⊟ functions must agree for all positive integer arguments.

# 16

# The Representation of Numbers

## 16.1 INTRODUCTION

A number can be represented in a variety of ways. For example, one dozen could be represented in any of the following three ways:

| | |
|---|---|
| \|\|\|\|\|\|\|\|\|\|\|\| | A list of one dozen marks. |
| XII | Roman numerals. |
| 12 | Decimal. |

There are many other useful ways of representing numbers, and no one of them is best, since each possesses advantages for certain purposes. For example, the first method above is much simpler for a beginner to understand than the decimal representation since the use of decimal representation requires an understanding of the notion of place value and of multiplying the successive digits by various powers of ten. On the other hand, a list of marks would be tedious to write for a large number such as 210. The Roman numeral system is also relatively simple and reasonably concise (for example CCX for decimal 210), but is much more awkward than decimal when addition or multiplication are to be performed on the numbers represented. Try, for example, to square XCXLIV (<u>without</u> converting to and from decimal), and then try to state explicit rules for carrying out such multiplication.

Any scheme for representing numbers is usually referred to as a <u>number system</u>. Table 16.1 illustrates each of the types of number systems for representing positive integers which will be considered in this chapter. This table will be referred to throughout the chapter and a careful examination of it at this point will probably be helpful. The chapter includes some discussion of the representation of negative and rational numbers. ⊡1

One way to gain an understanding of an unfamiliar number system is to learn how to perform operations such as addition and multiplication within it. To make such processes clear it will be helpful to write functions which define them. For example, if

```
      V←1 0 0 0 0 0 0
      W←1 1 0 0 0 0 0
```

then (in the Prime Factor system of Table 16.1) the vector *V* represents the number 2 and the vector *W* represents the number 6. Moreover, the vector *X*←*V*+*W* represents the product of these two numbers, that is, 12. Thus:

```
      X←V+W
      X
2 1 0 0 0 0 0
```

which represents 12 as may be seen from Table 16.1.

A function for multiplication in the Prime Factor system can therefore be defined as follows:

```
    ∇ Z←A PFTIMES B
[1]   Z←A+B
    ∇
```

For example:

```
      P←0 1 0 0 0 0 0      Represents 3
      Q←0 0 1 0 0 0 0      Represents 5
      P PFTIMES Q
0 1 1 0 0 0 0              Represents 15
```

The number represented by *BF* in the system R1 of Table 16.1 is represented by 1 5 in system R4. Determining the representation of a number in R4 from its representation in R1 is referred to as <u>converting</u> the number from R1 to R4.

Studying the rules for conversions from and to an unfamiliar number system provides a second approach to understanding it. In order to provide an example of defining the process of such a conversion it will first be convenient to introduce a new primitive function called <u>index-of</u>.

<u>Index-of</u>. The index of the quantity 7 in the vector

```
      V←2 5 7 1 8
```

is 3, that is, *V*[3] is 7. The symbol ι represents a dyadic function which yields the index of its right argument in its left argument. Thus:

```
      Vι7
3
      Vι1
4
```

| POSITIONAL SYSTEMS | | | | PUZZLES | | | | |
|---|---|---|---|---|---|---|---|---|
| Decimal | Binary | Octal | Prime Factors | R1 | R2 | R3 | R4 | R5 |
| 0 | 0 | 0 | | *A* | *N* | *Z* | 0 | 0 |
| 1 | 1 | 1 | 0 0 0 0 0 0 0 | *B* | *U* | *P* | 1 | 1 |
| 2 | 10 | 2 | 1 0 0 0 0 0 0 | *C* | *M* | *PN* | 2 | 1 0 |
| 3 | 11 | 3 | 0 1 0 0 0 0 0 | *D* | *B* | *PZ* | 3 | 1 1 |
| 4 | 100 | 4 | 2 0 0 0 0 0 0 | *E* | *E* | *PP* | 4 | 1 0 0 |
| 5 | 101 | 5 | 0 0 1 0 0 0 0 | *F* | *R* | *PNN* | 5 | 1 0 1 |
| 6 | 110 | 6 | 1 1 0 0 0 0 0 | *G* | *UN* | *PNZ* | 6 | 1 1 0 |
| 7 | 111 | 7 | 0 0 0 1 0 0 0 | *H* | *UU* | *PNP* | 7 | 1 1 1 |
| 8 | 1000 | 10 | 3 0 0 0 0 0 0 | *I* | *UM* | *PZN* | 8 | 1 0 0 0 |
| 9 | 1001 | 11 | 0 2 0 0 0 0 0 | *J* | *UB* | *PZZ* | 9 | 1 0 0 1 |
| 10 | 1010 | 12 | 1 0 1 0 0 0 0 | *BA* | *UE* | *PZP* | 1 0 | 1 0 1 0 |
| 11 | 1011 | 13 | 0 0 0 0 1 0 0 | *BB* | *UR* | *PPN* | 1 1 | 1 0 1 1 |
| 12 | 1100 | 14 | 2 1 0 0 0 0 0 | *BC* | *MN* | *PPZ* | 1 2 | 1 1 0 0 |
| 13 | 1101 | 15 | 0 0 0 0 0 1 0 | *BD* | *MU* | *PPP* | 1 3 | 1 1 0 1 |
| 14 | 1110 | 16 | 1 0 0 1 0 0 0 | *BE* | *MM* | *PNNN* | 1 4 | 1 1 1 0 |
| 15 | 1111 | 17 | 0 1 1 0 0 0 0 | *BF* | *MB* | *PNNZ* | 1 5 | 1 1 1 1 |
| 16 | 10000 | 20 | 4 0 0 0 0 0 0 | *BG* | *ME* | *PNNP* | 1 6 | 1 0 0 0 0 |
| 17 | 10001 | 21 | 0 0 0 0 0 0 1 | *BH* | *MR* | *PNZN* | 1 7 | 1 0 0 0 1 |

Various Representations of the First Eighteen Non-Negative Integers

Table 16.1

If the right argument is itself a vector, the result is the vector of the indices of each of its elements. For example:

```
      V⍳7 1
3 4
      2 3 5 7 11 ⍳ 5 11 3
3  5  2
      A←'ABCDEFGHIJKLMNOPQRSTUVWXYZ '
      J←A⍳'I SING OF OLAF'
      J
9  27  19  9  14  7  27  15  6  27  15  12  1  6
      A[J]
I SING OF OLAF
```

If W←3 1 4 3 7, then the index of 3 in W could be either 1 or 4; the ⍳ function is defined to yield the smallest possible value of the index. Thus:

```
      W⍳3
1
      8 6 10 4 6 2 10⍳2 4 6 8 10
6  4  2  1  3
```

If X is a value which does not occur in V, then V⍳X yields 1+⍴V, that is, one greater than the largest index of V:

```
      V←2 5 7 1 8
      V⍳4
6
      V⍳1 2 3 4 5 6 7 8
3 4  1  6  6  2  6  3  5
```

The conversion from system R1 to system R4 of Table 16.1 can be expressed simply in terms of the index-of function:

```
    ∇ Z←TO4 X
[1]   Z←¯1+'ABCDEFGHIJ'⍳X
    ∇
```

The inverse function for converting from R4 to R1 can be defined as follows:

```
    ∇Z←TO1 X
[1]   Z←'ABCDEFGHIJ'[1+X]
    ∇
```

For example:

```
      TO4 'BC'
1  2
      TO1 1 2
4 BC
```

We normally refer to any particular number by giving its decimal representation. Certain numbers are referred to in other ways (e.g., a dozen for 12, a gross for 144, and a score for 20), but most people would find it awkward to refer to or think of numbers such as 548 or 1247 in any terms other than their decimal representations. This poses a difficulty in discussing the decimal representation since it may be unclear as to whether a given sequence of digits refers to some number to be represented or to its representation in decimal. For example, it makes sense to say that the representation of the number 630 in Prime Factors is 1 2 1 1 0 0 0, but it seems fatuous to say that the representation of 630 in decimal is 630.

It would seem natural to begin the discussion of individual number systems with the decimal system since it is the most important and the most familiar. However, because of the matter discussed in the preceding paragraph we will begin instead with the less familiar Prime Factors system.

## 16.2 THE PRIME FACTORS SYSTEM

The scheme used in the Prime Factors number system should be clear from Exercise 7.20, and from Exercise 16.2, but will be summarized here. Any non-negative-integer vector $E$ (i.e., a vector whose elements are all non-negative integers) represents the number whose value is

```
      ×/P*E
```

where $P$ is the vector of the first $\rho E$ primes. For example, if $E$←2 0 2 0 1, then $P$←2 3 5 7 11 and

```
      P*E
4  1  25  1  11
      ×/P*E
1100
```

The system is most convenient for determining the product, greatest common divisor, or least common multiple of a pair of numbers.

Products. If

```
      M←×/P*E
      N←×/P*F
```

then the product $M{\times}N$ is equal to

```
      (×/P*E)×(×/P*F)
      ×/(P*E)×(P*F)    Commutativity and associativity of ×
      ×/P*E+F          Theorem 6 Section 14.7
```

Consequently the representation of $M{\times}N$ is simply $E+F$.

The foregoing arguments justify the function *PFTIMES* of Section 16.1. However, this function works only if the arguments *E* and *F* have the same number of elements. This limitation can be removed by first appending zeros to the shorter vector to make them both the same length as follows:

```
    ∇ Z←E PFT F
[1]   E←((ρE)⌈ρF)↑E
[2]   F←((ρE)⌈ρF)↑F
[3]   Z←E+F
    ∇
```

Appending zeros to the right of a vector makes no change in the number it represents. For example if *E*←2 0 2 1, then the corresponding value of *P* is 2 3 5 7 and

```
      ×/2 3 5 7*2 0 2 1
700
```

If *E*←2 0 2 1 0 0, then its corresponding value of *P* is 2 3 5 7 11 13 and

```
      ×/2 3 5 7 11 13*2 0 2 1 0 0
700
```

Since representations of the same length can be obtained so easily we will henceforth assume that all are of the same length.

<u>Greatest Common Divisor</u>. The greatest common divisor (defined in Section 12.3) is easily obtained in the Prime Factors system as follows:

```
    ∇ Z←E PFGCD F
[1]   Z←E⌊F
    ∇
```

For example, the representations of 24 and 54 are 3 1 and 1 3 and:

```
      1 3 PFGCD 3 1
1 1
      ×/2 3*1 3 PFGCD 3 1
6
```

The justification for the function *PFGCD* follows. If *M* is any integer and *DM* is any divisor of *M*, and if *E* and *DE* are the prime factor representations of *M* and *DM*, respectively (that is, *M*←×/*P*⋆*E* and *DM*←×/*P*⋆*DE*), then *DE* cannot exceed *E* in any component. That is, ∧/*DE*≤*E*. For suppose that *DE*[*K*]>*E*[*K*]. Then *P*[*K*]⋆*DE*[*K*] is a factor of the divisor of *M* but is not a factor of *M* itself. This is impossible.

Similarly, if $N \leftarrow \times / P \star F$ and $DN \leftarrow \times / P \star DF$ and $DN$ is a divisor of $N$, then $\wedge / DF \leq F$. Any common divisor of $M$ and $N$ must satisfy both these conditions. Therefore if $H$ is the representation of a common divisor, then $\wedge / H \leq E$ and $\wedge / H \leq F$.

The representation of the greatest common divisor is the largest vector satisfying both conditions and is therefore equal to $E \lfloor F$.

Least Common Multiple. A common multiple of two numbers is a multiple of each of the numbers, that is, each is a divisor of the common multiple. The least common multiple is the smallest of such common multiples. For example the first ten multiples of 24 and 54 are:

```
      24×⍳10
24  48  72  96  120  144  168  192  216  240
      54×⍳10
54  108  162  216  270  324  378  432  486  540
```

The least common multiple is therefore 216.

The function for least common multiple in the Prime Factor system follows:

```
    ∇ Z←E PFLCM F
[1]   Z←E⌈F
    ∇
```

Its justification is similar to that of the function *PFGCD* and is left as an exercise.

Factor Tables. Since multiplication, greatest common divisor, and least common multiple are so easy to compute in the Prime Factor system, a table of the Prime Factor representations of numbers can be useful for evaluating such functions. The fourth column of Table 16.1 provides the beginning of such a table, but its extension would soon require the tabulation of very long vectors. For example, there are 168 primes less than 1000 and a table to 1000 would therefore require some vectors of more than 160 elements.

A more compact scheme is used in giving the prime factorization in Tables 16.2 and 16.3 at the end of this chapter. Vectors of a fixed length (10) are used for the powers of the primes up to 29. Any other single prime which occurs as a factor is listed in a separate column. This scheme permits the representation of all positive integers up to but not including the square of the first prime (that is, 31) not covered by the fixed length vectors.

To illustrate the use of Table 16.2, consider the following problem: Find the greatest common divisor of $M \leftarrow 360$

and *N*←420. The corresponding representations *E* and *F* can be found from the table:

```
      E←3 2 1 0 0 0 0 0 0 0
      F←2 1 1 1 0 0 0 0 0 0
      E⌊F
2  1  1  0  0  0  0  0  0  0
```

The last vector above is the Prime Factor representation of the desired result. If one wants the normal decimal representation, it may be computed as follows:

```
      P←2 3 5 7 11 13 17 19 23 29
      ×/P*E⌊F
60
```

Alternatively, the decimal representation can be found by locating the vector *E*⌊*F* in Table 16.2 and reading off the decimal equivalent. However, it will be found to be very difficult to locate a given vector in the table because the vectors do not occur in any obvious sequence. For example, try to locate the vector 2 1 1 0 0 1 0 0 0 0 in the table without first evaluating its decimal representation. In order to make such use of the factorization table convenient, a re-ordered version of it is given in Table 16.3. In this table it is easy to locate the vector 2 1 1 0 0 1 0 0 0 0 by first scanning down to the <u>twos</u> in
5-9⊡ column 1, then down to the <u>ones</u> in column 2, and so on.

## 16.3 THE DECIMAL SYSTEM

The system R4 of Table 16.1 is closely related to the decimal system, for in it the successive digits of the decimal representation appear as the successive elements of a vector representation. This representation will be called the <u>vector decimal</u> system. It will be used to discuss the decimal system for two reasons:

1. The individual digits of the representations can be conveniently referred to by indexing.

2. The confusion between references to the number and its representation discussed in Section 16.1 can be avoided by its use.

The value of an integer expressed in ordinary decimal can be obtained from the vector decimal representation by the following function:

```
    ∇ Z←VDVAL X
[1]   Z←+/X×10*⌽¯1+⍳⍴,X
    ∇
```

For example:

```
      VDVAL 2 1 4
214
      VDVAL 0 0 2 1 4
214
      VDVAL 2 1 4 0 0
21400
```

The comma preceding the *X* in line 1 of the foregoing function ensures that the function ρ yields the number of elements in *X* even if *X* itself is a scalar rather than a vector. This matter is discussed in Sections 8.7 and 8.8.

From the definition of *VDVAL* it is clear that the elements of the vector representation *X* are multiplied by successive powers of 10 beginning with 10⋆0 (that is, 1) at the right. For example:

```
      X←2 1 4
      ¯1+ιρ,X
0  1  2
      ⌽¯1+ιρ,X
2  1  0
      10⋆⌽¯1+ιρ,X
100  10  1
      X×10⋆⌽¯1+ιρ,X
200  10  4
      +/X×10⋆⌽¯1+ιρ,X
214
```

The inverse function can be defined as follows:

```
     ∇ Z←IVDVAL X
[1]    Z←ι0
[2]    Z←(10|X),Z
[3]    X←(X-10|X)÷10
[4]    →2×X≠0
     ∇

      IVDVAL 214
2  1  4
```
⌹10

If zeros are appended to the left of a vector *D* then the number it represents in the vector decimal system is unchanged. For example, 0 0 2 1 4 represents the same number as does 2 1 4. One vector can therefore be easily extended by zeros to make it the same dimension as some other vector. We will therefore assume that the vectors discussed are already of equal dimensions.

<u>Addition</u>. If *A*←3 1 4 and *B*←4 7 2 are the vector decimal representations of two numbers, then the representation of their sum is obtained very simply as follows:

```
      A+B
7  8  6
      VDVAL A                             VDVAL B
314                              472
      (VDVAL A)+VDVAL B                   VDVAL A+B
786                              786
```

The reason why this addition works is clear; digits to be multiplied by the same power of ten are summed to yield the result.

If *A*←3 6 4 and *B*←4 7 2, the same process works in the sense that *VDVAL A+B* yields the same result as (*VDVAL A*)+(*VDVAL B*). Thus

```
      A+B
7  13  6
      VDVAL A+B
836
      (VDVAL A)+VDVAL B
836
```

However, 7 13 6 would not be accepted as a suitable representation of the number 836, since we normally require
11⍠ that each of the digits be less than 10.

A digit larger than 10 can be eliminated by "carrying" any multiple of ten to the next position to the left (whose weight is ten times that of the column it precedes). A function for addition based on this notion can be defined as follows:

```
     ∇ Z←A VDADD B
[1]    Z←10|A+B
[2]    C←(1↓10≤A+B),0
[3]    →4×∨/0≠C
[4]    A←Z
[5]    B←C
[6]    →1
     ∇
```

A trace of the foregoing function for the arguments 3 7 9 and 2 2 1 will show why the statements 2 to 5 may have to be repeated several times:

```
      3 7 9 VDADD 2 2 1
VDADD[1] 5  9  0
VDADD[2] 0  1  0
VDADD[1] 5  0  0
VDADD[2] 1  0  0
VDADD[1] 6  0  0
VDADD[2] 0  0  0
6  0  0
```

This trace also reveals that the method is <u>not</u> the method normally learned for addition but that it is a reasonably satisfactory method for manually adding two numbers.

The more familiar procedure for addition is described by the following function:

```
    ∇ Z←A SERIALDADD B
[1]   Z←⍳0
[2]   C←0
[3]   I←1+⍴B
[4]   I←I-1
[5]   →6×I≠0
[6]   N←C+A[I]+B[I]
[7]   Z←(10|N),Z
[8]   C←10≤N
[9]   →4
    ∇
```

For example:

```
      T∆SERIALDADD←6 7 8
      3 7 9 SERIALDADD 2 2 1
SERIALDADD[6] 10
SERIALDADD[7] 0
SERIALDADD[8] 1
SERIALDADD[6] 10
SERIALDADD[7] 0  0
SERIALDADD[8] 1
SERIALDADD[6] 6
SERIALDADD[7] 6  0  0
SERIALDADD[8] 0
6  0  0
```

⌹12-13

## 16.4 THE BINARY SYSTEM

The binary number system is illustrated in column 2 of Table 16.1, and the corresponding vector binary system is illustrated in the last column. It is similar to the decimal system, differing only in that the weights applied to the digit positions are powers of two rather than ten, and that the digit values are limited to less than 2, that is, to the values 0 and 1.

Functions for treating numbers represented in vector binary can therefore be derived from the corresponding functions for vector decimal by simply replacing each occurrence of 10 by 2:

```
     ∇ Z←VDVAL X
[1]    Z←+/X×10*⌽¯1+ιρ,X∇

     ∇ Z←IVDVAL X
[1]    Z←ι0
[2]    Z←(10|X),Z
[3]    X←(X-10|X)÷10
[4]    →2×X≠0∇
```

```
     ∇ Z←VBVAL X
[1]    Z←+/X×2*⌽¯1+ιρ,X∇

     ∇ Z←IVBVAL X
[1]    Z←ι0
[2]    Z←(2|X),Z
[3]    X←(X-2|X)÷2
[4]    →2×X≠0∇
```

For example:

```
       Q←1 1 0 1 0
       R←VBVAL Q
       R
26
       IVBVAL R
1  1  0  1  0
       IVDVAL VBVAL Q
2  6
       IVBVAL VDVAL 2 6
1  1  0  1  0
```

14-18⌹

## 16.5 POSITIONAL NUMBER SYSTEMS

The decimal and binary systems discussed in the preceding sections are examples of *positional number systems*; any positive integer greater than 1 could be used instead of 2 or 10 and the number chosen to play that role is called the *base* of the number system. If the number is *N*, the system is referred to as a *base-N* number system, although other names such as *binary* (for 2), *decimal* (for 10), *octal* (for 8) and *ternary* (for 3) are often used.

General functions for treating any positional system could be obtained by substituting the name *BASE* for every occurrence of 10 in the functions of Section 16.3 and then assigning the desired value to *BASE*. For example, a ternary system could be treated as follows:

```
     ∇ Z←VAL X
[1]    Z←+/X×BASE*⌽¯1+ιρ,X∇

       BASE←3
       VAL 1 0
3
       VAL 1 1
4
       VAL 2 1 1
22
```

The function *VAL* defined above for evaluating a positional number system has a close and interesting relation to the polynomial function *POLY* defined in Section 13.5 and repeated below:

```
    ∇ Z←C POLY X
[1]   Z←+/C×X*¯1+⍳⍴,C
    ∇
```

For example:

```
      BASE←10
      Q←3 1 4
      VAL Q
314
      (⌽Q) POLY BASE
314
      BASE←8
      VAL Q
204
      (⌽Q) POLY BASE
204
```

From these examples (and from the similarities in the definitions of the functions *VAL* and *POLY*) it should be clear that the value function *VAL* is in effect a polynomial whose argument is the value of the base and whose coefficients are reversed in order, i.e., the powers of the base are in descending rather than ascending order. ⌹19

The digits used in a positional number system normally run from zero to one less than the base, i.e., the digits for base *B* are chosen from the vector ¯1+⍳*B*. This limitation is not essential, and useful representations can be formed using a different restriction. For example, in a ternary system the digits might be limited to ¯1, 0, and 1 rather than 0, 1 and 2. Thus:

```
      BASE←3
      VAL 0
0
      VAL 1
1
      VAL 1 ¯1
2
      VAL 1 0
3
      VAL 1 1
4
      VAL 1 ¯1 ¯1
5
```

```
      VAL ¯1
¯1
      VAL ¯1 1
¯2
      VAL ¯1 0
¯3
      VAL ¯1 ¯1
¯4
      VAL ¯1 1 1
¯5
```

From the foregoing it is clear that this system can represent negative as well as positive integers. ⌹20

## 16.6 ADDITION AND MULTIPLICATION TABLES

In learning to add decimal numbers one must learn not only the rules for handling carries but also the sums of all the digit pairs, in other words one must learn the addition table for the arguments zero to nine. The body of this table appears as follows:

```
      S←0 1 2 3 4 5 6 7 8 9
      A←S∘.+S
      A
 0   1   2   3   4   5   6   7   8   9
 1   2   3   4   5   6   7   8   9  10
 2   3   4   5   6   7   8   9  10  11
 3   4   5   6   7   8   9  10  11  12
 4   5   6   7   8   9  10  11  12  13
 5   6   7   8   9  10  11  12  13  14
 6   7   8   9  10  11  12  13  14  15
 7   8   9  10  11  12  13  14  15  16
 8   9  10  11  12  13  14  15  16  17
 9  10  11  12  13  14  15  16  17  18
```

Strictly speaking, one does not use the addition table directly but rather uses the following "sum" and "carry" tables derived from it:

```
      10|A                        10≤A
 0 1 2 3 4 5 6 7 8 9          0 0 0 0 0 0 0 0 0 0
 1 2 3 4 5 6 7 8 9 0          0 0 0 0 0 0 0 0 0 1
 2 3 4 5 6 7 8 9 0 1          0 0 0 0 0 0 0 0 1 1
 3 4 5 6 7 8 9 0 1 2          0 0 0 0 0 0 0 1 1 1
 4 5 6 7 8 9 0 1 2 3          0 0 0 0 0 0 1 1 1 1
 5 6 7 8 9 0 1 2 3 4          0 0 0 0 0 1 1 1 1 1
 6 7 8 9 0 1 2 3 4 5          0 0 0 0 1 1 1 1 1 1
 7 8 9 0 1 2 3 4 5 6          0 0 0 1 1 1 1 1 1 1
 8 9 0 1 2 3 4 5 6 7          0 0 1 1 1 1 1 1 1 1
 9 0 1 2 3 4 5 6 7 8          0 1 1 1 1 1 1 1 1 1
```

In order to perform addition in a system with a base other than ten it is helpful to construct the corresponding tables for that base:

```
      S←0 1 2 3 4 5 6 7
      A←S∘.+S
   A                          8|A                   8≤A
0  1  2  3  4  5  6  7    0 1 2 3 4 5 6 7    0 0 0 0 0 0 0 0
1  2  3  4  5  6  7  8    1 2 3 4 5 6 7 0    0 0 0 0 0 0 0 1
2  3  4  5  6  7  8  9    2 3 4 5 6 7 0 1    0 0 0 0 0 0 1 1
3  4  5  6  7  8  9 10    3 4 5 6 7 0 1 2    0 0 0 0 0 1 1 1
4  5  6  7  8  9 10 11    4 5 6 7 0 1 2 3    0 0 0 0 1 1 1 1
5  6  7  8  9 10 11 12    5 6 7 0 1 2 3 4    0 0 0 1 1 1 1 1
6  7  8  9 10 11 12 13    6 7 0 1 2 3 4 5    0 0 1 1 1 1 1 1
7  8  9 10 11 12 13 14    7 0 1 2 3 4 5 6    0 1 1 1 1 1 1 1
```

```
      S←0 1
      A←S∘.+S
      A                   2|A                  2≤A
 0 1                 0 1                  0 0
 1 2                 1 0                  0 1
```

Similar remarks apply to multiplication. Appropriate tables for base 8 are shown below:

```
      S←0 1 2 3 4 5 6 7
      M←S∘.×S
      M                          8|M                 ⌊M÷8
0  0  0  0  0  0  0  0     0 0 0 0 0 0 0 0     0 0 0 0 0 0 0 0
0  1  2  3  4  5  6  7     0 1 2 3 4 5 6 7     0 0 0 0 0 0 0 0
0  2  4  6  8 10 12 14     0 2 4 6 0 2 4 6     0 0 0 0 1 1 1 1
0  3  6  9 12 15 18 21     0 3 6 1 4 7 2 5     0 0 0 1 1 1 2 2
0  4  8 12 16 20 24 28     0 4 0 4 0 4 0 4     0 0 1 1 2 2 3 3
0  5 10 15 20 25 30 35     0 5 2 7 4 1 6 3     0 0 1 1 2 3 3 4
0  6 12 18 24 30 36 42     0 6 4 2 0 6 4 2     0 0 1 2 3 3 4 5
0  7 14 21 28 35 42 49     0 7 6 5 4 3 2 1     0 0 1 2 3 4 5 6
```

Since multiplication may produce a carry greater than 1 it is necessary to replace the simple expression `8≤M` by the expression `⌊M÷8` as shown for the carry table above. ⌹21-22

## 16.7 NEGATIVE INTEGERS

Although the last system discussed in Section 16.5 can represent negative as well as positive integers, the common positional number systems (such as the decimal) can represent non-negative integers only, and negative integers are represented by an extra character (called a negative sign) appended to the left of the digit. In the corresponding vector representations (such as vector decimal and vector binary) it is necessary to add a component to the vector to represent the sign. For example, if the first position is used to represent the sign and if 0 represents positive and 1 represents negative, then the following function serves to evaluate such a vector binary representation:

```
      ∇ Z←NVBVAL X
[1]     Z←(¯1*X[1])×+/(1↓X)×2*⌽¯1+⍳⍴1↓X
      ∇
```

Moreover, the representations of the integers from ¯5 to 5 in this system would appear as follows:

| | |
|---|---|
| ¯5 | 1 1 0 1 |
| ¯4 | 1 1 0 0 |
| ¯3 | 1 0 1 1 |
| ¯2 | 1 0 1 0 |
| ¯1 | 1 0 0 1 |
| 0 | 0 0 0 0 |
| 1 | 0 0 0 1 |
| 2 | 0 0 1 0 |
| 3 | 0 0 1 1 |
| 4 | 0 1 0 0 |
| 5 | 0 1 0 1 |

⌹23

16.8 RATIONAL NUMBERS

The scheme of representing a rational number by a two-element vector introduced in Section 5.7 can be considered as a number system for representing rational numbers and will be called the rational vector or RV system. Functions for addition and multiplication in the RV system were treated in Section 9.5. The decimal value of such a vector *V* in the RV system is, of course, the simple expression ÷/*V*.

We will now consider the conversion of decimal fractions and repeating decimal fractions to the RV system. The number 24.361 is represented in RV as 24361 1000. In general, the second element of the RV representation is 10⋆*N*, where *N* is the number of digits following the decimal point. To clarify these matters we will introduce a system called fraction vector decimal (FVD) in which the first element specifies the number of digits following the decimal point, and the remaining elements are the usual vector decimal representation. For example:

| Decimal | FVD | RV |
|---|---|---|
| 24.361 | 3 2 4 3 6 1 | 24361 1000 |
| .0024 | 4 0 0 2 4 | 24 10000 |

The following function converts from FVD to RV:

```
    ∇ Z←FVD X
[1]   Z←(VDVAL 1↓X),10⋆X[1]∇
```

For example:

```
      FVD 3 2 4 3 6 1
24361  1000
```

24-25⌹ The function *VDVAL* is defined in Section 16.3.

Certain rationals cannot be expressed exactly as decimal fractions because the division involved produces a never-ending sequence of digits. The first twenty digits of such sequences are shown below for a few cases:

```
   7÷3        2.3333333333333333333
 239÷11      21.727272727272727272
 227÷70       3.2428571428571428571
```

In each of these examples some pattern of digits is soon established which repeats unendingly. The first case shows a pattern of a single digit (3), the second a pattern of two (72), and the third a pattern of six (428571).

The repetition is no accident; any case which does not terminate must show a pattern. For, if *D* is the divisor, then no more than *D* different remainders can arise in the division process and one such remainder must recur. When a remainder recurs, the same sequence of quotients must also recur.

The determination of the rational (RV) representation of a number from its repeating decimal form will now be illustrated for the number 227÷70. We attempt to make the second element of the rational representation (i.e., the denominator) equal to (10*6)-1, because the length of the repeating pattern is 6. Then the first element must be (10*6)-1 times the actual number *N*. Thus:

```
    N×10*6 : 3242857.142857142857142857142∘∘∘
    N× 1   :       3.242857142857142∘∘∘
Difference : 3242853.9
```

Since *N*×10*6 is obtained from *N* by moving the decimal point to the right by a distance equal to the length of the repeating pattern, it is clear that the difference (*N*×10*6)-*N*×1 will be a terminating decimal. However, this difference is still not an integer and it is necessary to multiply both it and the intended denominator (that is, (10*6)-1) by ten. Thus the desired representation consists of the two elements:

```
      32428539 ((10*6)-1)×10
```

or

```
      32428539 9999990
```

Thus *V*←32428539 9999990 is the RV representation of 227÷70. Clearly *W*←227 70 is also an RV representation of the same number and the two can be shown to be equivalent as follows:

```
      V÷W
142857  142857
```

In order to show the details of this process for a general case we will now adopt a system for representing repeating decimals. A repeating decimal can be specified exactly by the following three pieces of information:

1. The sequence of digits up to any point which includes the complete repeating pattern.

2. The number of digits following the decimal point.

3. The length of the repeating pattern.

Repeating decimals can therefore be represented in a <u>repeating fraction vector decimal</u> (RFVD) system obtained by prefixing the FVD representation by an element specifying the length of the repeating pattern. For example:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 7÷3 | 1 | 1 | 2 | 3 | | | | | | | | |
| 239÷11 | 2 | 2 | 2 | 1 | 7 | 2 | | | | | | |
| 227÷70 | 6 | 8 | 3 | 2 | 4 | 2 | 8 | 5 | 7 | 1 | 4 | |
| 227÷70 | 6 | 9 | 3 | 2 | 4 | 2 | 8 | 5 | 7 | 1 | 4 | 2 |

The two representations for 227÷70 illustrate that the RFVD representation is not unique; the sequence of digits may be cut off at any point which includes the complete repeating pattern.

The process illustrated for the case 227÷70 can now be specified more generally by the following function for converting from RFVD to RV:

```
    ∇ Z←RFVD X
[1]   Z←(VDVAL(2↓X),(-X[1])↑X)-VDVAL 2↓X
[2]   Z←Z,(10*X[2])×¯1+10*X[1]∇
```

For example:

```
      V←6 8 3 2 4 2 8 5 7 1 4
      Q←RFVD V
      Q
324285390000000  99999900000000
```

It is clear that the function *RFVD* (like the process used in the earlier example) does not yield the smallest possible values for the result *Q*. To obtain the representation in the reduced form it is necessary to divide the elements of *Q* by any factor common to them, i.e., to divide by the greatest common divisor of the elements of *Q*. For this purpose we will recall and use the *GCD* function of Section 12.3:

```
    ∇ Z←GCD X
[1]   Z←X[1]
[2]   X←(|/X),X[1]
[3]   →X[1]≠0∇
```

26-27⊡

```
      GCD Q
1428570000000
      Q÷GCD Q
227  70
```

# Factorization
# Tables

| | 1 | 11122 | | | 1 | 11122 | | | 1 | 11122 | | | 1 | 11122 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 23571 | 37939 | | | 23571 | 37939 | | | 23571 | 37939 | | | 23571 | 37939 | |
| 1 | 00000 | 00000 | | 61* | 00000 | 00000 | 61 | 121 | 00002 | 00000 | | 181* | 00000 | 00000 | 181 |
| 2* | 10000 | 00000 | | 62 | 10000 | 00000 | 31 | 122 | 10000 | 00000 | 61 | 182 | 10010 | 10000 | |
| 3* | 01000 | 00000 | | 63 | 02010 | 00000 | | 123 | 01000 | 00000 | 41 | 183 | 01000 | 00000 | 61 |
| 4 | 20000 | 00000 | | 64 | 60000 | 00000 | | 124 | 20000 | 00000 | 31 | 184 | 30000 | 00010 | |
| 5* | 00100 | 00000 | | 65 | 00100 | 10000 | | 125 | 00300 | 00000 | | 185 | 00100 | 00000 | 37 |
| 6 | 11000 | 00000 | | 66 | 11001 | 00000 | | 126 | 12010 | 00000 | | 186 | 11000 | 00000 | 31 |
| 7* | 00010 | 00000 | | 67* | 00000 | 00000 | 67 | 127* | 00000 | 00000 | 127 | 187 | 00001 | 01000 | |
| 8 | 30000 | 00000 | | 68 | 20000 | 01000 | | 128 | 70000 | 00000 | | 188 | 20000 | 00000 | 47 |
| 9 | 02000 | 00000 | | 69 | 01000 | 00010 | | 129 | 01000 | 00000 | 43 | 189 | 03010 | 00000 | |
| 10 | 10100 | 00000 | | 70 | 10110 | 00000 | | 130 | 10100 | 10000 | | 190 | 10100 | 00100 | |
| 11* | 00001 | 00000 | | 71* | 00000 | 00000 | 71 | 131* | 00000 | 00000 | 131 | 191* | 00000 | 00000 | 191 |
| 12 | 21000 | 00000 | | 72 | 32000 | 00000 | | 132 | 21001 | 00000 | | 192 | 61000 | 00000 | |
| 13* | 00000 | 10000 | | 73* | 00000 | 00000 | 73 | 133 | 00010 | 00100 | | 193* | 00000 | 00000 | 193 |
| 14 | 10010 | 00000 | | 74 | 10000 | 00000 | 37 | 134 | 10000 | 00000 | 67 | 194 | 10000 | 00000 | 97 |
| 15 | 01100 | 00000 | | 75 | 01200 | 00000 | | 135 | 03100 | 00000 | | 195 | 01100 | 10000 | |
| 16 | 40000 | 00000 | | 76 | 20000 | 00100 | | 136 | 30000 | 01000 | | 196 | 20020 | 00000 | |
| 17* | 00000 | 01000 | | 77 | 00011 | 00000 | | 137* | 00000 | 00000 | 137 | 197* | 00000 | 00000 | 197 |
| 18 | 12000 | 00000 | | 78 | 11000 | 10000 | | 138 | 11000 | 00010 | | 198 | 12001 | 00000 | |
| 19* | 00000 | 00100 | | 79* | 00000 | 00000 | 79 | 139* | 00000 | 00000 | 139 | 199* | 00000 | 00000 | 199 |
| 20 | 20100 | 00000 | | 80 | 40100 | 00000 | | 140 | 20110 | 00000 | | 200 | 30200 | 00000 | |
| 21 | 01010 | 00000 | | 81 | 04000 | 00000 | | 141 | 01000 | 00000 | 47 | 201 | 01000 | 00000 | 67 |
| 22 | 10001 | 00000 | | 82 | 10000 | 00000 | 41 | 142 | 10000 | 00000 | 71 | 202 | 10000 | 00000 | 101 |
| 23* | 00000 | 00010 | | 83* | 00000 | 00000 | 83 | 143 | 00001 | 10000 | | 203 | 00010 | 00001 | |
| 24 | 31000 | 00000 | | 84 | 21010 | 00000 | | 144 | 42000 | 00000 | | 204 | 21000 | 01000 | |
| 25 | 00200 | 00000 | | 85 | 00100 | 01000 | | 145 | 00100 | 00001 | | 205 | 00100 | 00000 | 41 |
| 26 | 10000 | 10000 | | 86 | 10000 | 00000 | 43 | 146 | 10000 | 00000 | 73 | 206 | 10000 | 00000 | 103 |
| 27 | 03000 | 00000 | | 87 | 01000 | 00001 | | 147 | 01020 | 00000 | | 207 | 02000 | 00010 | |
| 28 | 20010 | 00000 | | 88 | 30001 | 00000 | | 148 | 20000 | 00000 | 37 | 208 | 40000 | 10000 | |
| 29* | 00000 | 00001 | | 89* | 00000 | 00000 | 89 | 149* | 00000 | 00000 | 149 | 209 | 00001 | 00100 | |
| 30 | 11100 | 00000 | | 90 | 12100 | 00000 | | 150 | 11200 | 00000 | | 210 | 11110 | 00000 | |
| 31* | 00000 | 00000 | 31 | 91 | 00010 | 10000 | | 151* | 00000 | 00000 | 151 | 211* | 00000 | 00000 | 211 |
| 32 | 50000 | 00000 | | 92 | 20000 | 00010 | | 152 | 30000 | 00100 | | 212 | 20000 | 00000 | 53 |
| 33 | 01001 | 00000 | | 93 | 01000 | 00000 | 31 | 153 | 02000 | 01000 | | 213 | 01000 | 00000 | 71 |
| 34 | 10000 | 01000 | | 94 | 10000 | 00000 | 47 | 154 | 10011 | 00000 | | 214 | 10000 | 00000 | 107 |
| 35 | 00110 | 00000 | | 95 | 00100 | 00100 | | 155 | 00100 | 00000 | 31 | 215 | 00100 | 00000 | 43 |
| 36 | 22000 | 00000 | | 96 | 51000 | 00000 | | 156 | 21000 | 10000 | | 216 | 33000 | 00000 | |
| 37* | 00000 | 00000 | 37 | 97* | 00000 | 00000 | 97 | 157* | 00000 | 00000 | 157 | 217 | 00010 | 00000 | 31 |
| 38 | 10000 | 00100 | | 98 | 10020 | 00000 | | 158 | 10000 | 00000 | 79 | 218 | 10000 | 00000 | 109 |
| 39 | 01000 | 10000 | | 99 | 02001 | 00000 | | 159 | 01000 | 00000 | 53 | 219 | 01000 | 00000 | 73 |
| 40 | 30100 | 00000 | | 100 | 20200 | 00000 | | 160 | 50100 | 00000 | | 220 | 20101 | 00000 | |
| 41* | 00000 | 00000 | 41 | 101* | 00000 | 00000 | 101 | 161 | 00010 | 00010 | | 221 | 00000 | 11000 | |
| 42 | 11010 | 00000 | | 102 | 11000 | 01000 | | 162 | 14000 | 00000 | | 222 | 11000 | 00000 | 37 |
| 43* | 00000 | 00000 | 43 | 103* | 00000 | 00000 | 103 | 163* | 00000 | 00000 | 163 | 223* | 00000 | 00000 | 223 |
| 44 | 20001 | 00000 | | 104 | 30000 | 10000 | | 164 | 20000 | 00000 | 41 | 224 | 50010 | 00000 | |
| 45 | 02100 | 00000 | | 105 | 01110 | 00000 | | 165 | 01101 | 00000 | | 225 | 02200 | 00000 | |
| 46 | 10000 | 00010 | | 106 | 10000 | 00000 | 53 | 166 | 10000 | 00000 | 83 | 226 | 10000 | 00000 | 113 |
| 47* | 00000 | 00000 | 47 | 107* | 00000 | 00000 | 107 | 167* | 00000 | 00000 | 167 | 227* | 00000 | 00000 | 227 |
| 48 | 41000 | 00000 | | 108 | 23000 | 00000 | | 168 | 31010 | 00000 | | 228 | 21000 | 00100 | |
| 49 | 00020 | 00000 | | 109* | 00000 | 00000 | 109 | 169 | 00000 | 20000 | | 229* | 00000 | 00000 | 229 |
| 50 | 10200 | 00000 | | 110 | 10101 | 00000 | | 170 | 10100 | 01000 | | 230 | 10100 | 00010 | |
| 51 | 01000 | 01000 | | 111 | 01000 | 00000 | 37 | 171 | 02000 | 00100 | | 231 | 01011 | 00000 | |
| 52 | 20000 | 10000 | | 112 | 40010 | 00000 | | 172 | 20000 | 00000 | 43 | 232 | 30000 | 00001 | |
| 53* | 00000 | 00000 | 53 | 113* | 00000 | 00000 | 113 | 173* | 00000 | 00000 | 173 | 233* | 00000 | 00000 | 233 |
| 54 | 13000 | 00000 | | 114 | 11000 | 00100 | | 174 | 11000 | 00001 | | 234 | 12000 | 10000 | |
| 55 | 00101 | 00000 | | 115 | 00100 | 00010 | | 175 | 00210 | 00000 | | 235 | 00100 | 00000 | 47 |
| 56 | 30010 | 00000 | | 116 | 20000 | 00001 | | 176 | 40001 | 00000 | | 236 | 20000 | 00000 | 59 |
| 57 | 01000 | 00100 | | 117 | 02000 | 10000 | | 177 | 01000 | 00000 | 59 | 237 | 01000 | 00000 | 79 |
| 58 | 10000 | 00001 | | 118 | 10000 | 00000 | 59 | 178 | 10000 | 00000 | 89 | 238 | 10010 | 01000 | |
| 59* | 00000 | 00000 | 59 | 119 | 00010 | 01000 | | 179* | 00000 | 00000 | 179 | 239* | 00000 | 00000 | 239 |
| 60 | 21100 | 00000 | | 120 | 31100 | 00000 | | 180 | 22100 | 00000 | | 240 | 41100 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
| | 23571 | 37939 | |
| 241* | 00000 | 00000 | 241 |
| 242 | 10002 | 00000 | |
| 243 | 05000 | 00000 | |
| 244 | 20000 | 00000 | 61 |
| 245 | 00120 | 00000 | |
| 246 | 11000 | 00000 | 41 |
| 247 | 00000 | 10100 | |
| 248 | 30000 | 00000 | 31 |
| 249 | 01000 | 00000 | 83 |
| 250 | 10300 | 00000 | |
| 251* | 00000 | 00000 | 251 |
| 252 | 22010 | 00000 | |
| 253 | 00001 | 00010 | |
| 254 | 10000 | 00000 | 127 |
| 255 | 01100 | 01000 | |
| 256 | 80000 | 00000 | |
| 257* | 00000 | 00000 | 257 |
| 258 | 11000 | 00000 | 43 |
| 259 | 00010 | 00000 | 37 |
| 260 | 20100 | 10000 | |
| 261 | 02000 | 00001 | |
| 262 | 10000 | 00000 | 131 |
| 263* | 00000 | 00000 | 263 |
| 264 | 31001 | 00000 | |
| 265 | 00100 | 00000 | 53 |
| 266 | 10010 | 00100 | |
| 267 | 01000 | 00000 | 89 |
| 268 | 20000 | 00000 | 67 |
| 269* | 00000 | 00000 | 269 |
| 270 | 13100 | 00000 | |
| 271* | 00000 | 00000 | 271 |
| 272 | 40000 | 01000 | |
| 273 | 01010 | 10000 | |
| 274 | 10000 | 00000 | 137 |
| 275 | 00201 | 00000 | |
| 276 | 21000 | 00010 | |
| 277* | 00000 | 00000 | 277 |
| 278 | 10000 | 00000 | 139 |
| 279 | 02000 | 00000 | 31 |
| 280 | 30110 | 00000 | |
| 281* | 00000 | 00000 | 281 |
| 282 | 11000 | 00000 | 47 |
| 283* | 00000 | 00000 | 283 |
| 284 | 20000 | 00000 | 71 |
| 285 | 01100 | 00100 | |
| 286 | 10001 | 10000 | |
| 287 | 00010 | 00000 | 41 |
| 288 | 52000 | 00000 | |
| 289 | 00000 | 02000 | |
| 290 | 10100 | 00001 | |
| 291 | 01000 | 00000 | 97 |
| 292 | 20000 | 00000 | 73 |
| 293* | 00000 | 00000 | 293 |
| 294 | 11020 | 00000 | |
| 295 | 00100 | 00000 | 59 |
| 296 | 30000 | 00000 | 37 |
| 297 | 03001 | 00000 | |
| 298 | 10000 | 00000 | 149 |
| 299 | 00000 | 10010 | |
| 300 | 21200 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
| | 23571 | 37939 | |
| 301 | 00010 | 00000 | 43 |
| 302 | 10000 | 00000 | 151 |
| 303 | 01000 | 00000 | 101 |
| 304 | 40000 | 00100 | |
| 305 | 00100 | 00000 | 61 |
| 306 | 12000 | 01000 | |
| 307* | 00000 | 00000 | 307 |
| 308 | 20011 | 00000 | |
| 309 | 01000 | 00000 | 103 |
| 310 | 10100 | 00000 | 31 |
| 311* | 00000 | 00000 | 311 |
| 312 | 31000 | 10000 | |
| 313* | 00000 | 00000 | 313 |
| 314 | 10000 | 00000 | 157 |
| 315 | 02110 | 00000 | |
| 316 | 20000 | 00000 | 79 |
| 317* | 00000 | 00000 | 317 |
| 318 | 11000 | 00000 | 53 |
| 319 | 00001 | 00001 | |
| 320 | 60100 | 00000 | |
| 321 | 01000 | 00000 | 107 |
| 322 | 10010 | 00010 | |
| 323 | 00000 | 01100 | |
| 324 | 24000 | 00000 | |
| 325 | 00200 | 10000 | |
| 326 | 10000 | 00000 | 163 |
| 327 | 01000 | 00000 | 109 |
| 328 | 30000 | 00000 | 41 |
| 329 | 00010 | 00000 | 47 |
| 330 | 11101 | 00000 | |
| 331* | 00000 | 00000 | 331 |
| 332 | 20000 | 00000 | 83 |
| 333 | 02000 | 00000 | 37 |
| 334 | 10000 | 00000 | 167 |
| 335 | 00100 | 00000 | 67 |
| 336 | 41010 | 00000 | |
| 337* | 00000 | 00000 | 337 |
| 338 | 10000 | 20000 | |
| 339 | 01000 | 00000 | 113 |
| 340 | 20100 | 01000 | |
| 341 | 00001 | 00000 | 31 |
| 342 | 12000 | 00100 | |
| 343 | 00030 | 00000 | |
| 344 | 30000 | 00000 | 43 |
| 345 | 01100 | 00010 | |
| 346 | 10000 | 00000 | 173 |
| 347* | 00000 | 00000 | 347 |
| 348 | 21000 | 00001 | |
| 349* | 00000 | 00000 | 349 |
| 350 | 10210 | 00000 | |
| 351 | 03000 | 10000 | |
| 352 | 50001 | 00000 | |
| 353* | 00000 | 00000 | 353 |
| 354 | 11000 | 00000 | 59 |
| 355 | 00100 | 00000 | 71 |
| 356 | 20000 | 00000 | 89 |
| 357 | 01010 | 01000 | |
| 358 | 10000 | 00000 | 179 |
| 359* | 00000 | 00000 | 359 |
| 360 | 32100 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
| | 23571 | 37939 | |
| 361 | 00000 | 00200 | |
| 362 | 10000 | 00000 | 181 |
| 363 | 01002 | 00000 | |
| 364 | 20010 | 10000 | |
| 365 | 00100 | 00000 | 73 |
| 366 | 11000 | 00000 | 61 |
| 367* | 00000 | 00000 | 367 |
| 368 | 40000 | 00010 | |
| 369 | 02000 | 00000 | 41 |
| 370 | 10100 | 00000 | 37 |
| 371 | 00010 | 00000 | 53 |
| 372 | 21000 | 00000 | 31 |
| 373* | 00000 | 00000 | 373 |
| 374 | 10001 | 01000 | |
| 375 | 01300 | 00000 | |
| 376 | 30000 | 00000 | 47 |
| 377 | 00000 | 10001 | |
| 378 | 13010 | 00000 | |
| 379* | 00000 | 00000 | 379 |
| 380 | 20100 | 00100 | |
| 381 | 01000 | 00000 | 127 |
| 382 | 10000 | 00000 | 191 |
| 383* | 00000 | 00000 | 383 |
| 384 | 71000 | 00000 | |
| 385 | 00111 | 00000 | |
| 386 | 10000 | 00000 | 193 |
| 387 | 02000 | 00000 | 43 |
| 388 | 20000 | 00000 | 97 |
| 389* | 00000 | 00000 | 389 |
| 390 | 11100 | 10000 | |
| 391 | 00000 | 01010 | |
| 392 | 30020 | 00000 | |
| 393 | 01000 | 00000 | 131 |
| 394 | 10000 | 00000 | 197 |
| 395 | 00100 | 00000 | 79 |
| 396 | 22001 | 00000 | |
| 397* | 00000 | 00000 | 397 |
| 398 | 10000 | 00000 | 199 |
| 399 | 01010 | 00100 | |
| 400 | 40200 | 00000 | |
| 401* | 00000 | 00000 | 401 |
| 402 | 11000 | 00000 | 67 |
| 403 | 00000 | 10000 | 31 |
| 404 | 20000 | 00000 | 101 |
| 405 | 04100 | 00000 | |
| 406 | 10010 | 00001 | |
| 407 | 00001 | 00000 | 37 |
| 408 | 31000 | 01000 | |
| 409* | 00000 | 00000 | 409 |
| 410 | 10100 | 00000 | 41 |
| 411 | 01000 | 00000 | 137 |
| 412 | 20000 | 00000 | 103 |
| 413 | 00010 | 00000 | 59 |
| 414 | 12000 | 00010 | |
| 415 | 00100 | 00000 | 83 |
| 416 | 50000 | 10000 | |
| 417 | 01000 | 00000 | 139 |
| 418 | 10001 | 00100 | |
| 419* | 00000 | 00000 | 419 |
| 420 | 21110 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
| | 23571 | 37939 | |
| 421* | 00000 | 00000 | 421 |
| 422 | 10000 | 00000 | 211 |
| 423 | 02000 | 00000 | 47 |
| 424 | 30000 | 00000 | 53 |
| 425 | 00200 | 01000 | |
| 426 | 11000 | 00000 | 71 |
| 427 | 00010 | 00000 | 61 |
| 428 | 20000 | 00000 | 107 |
| 429 | 01001 | 10000 | |
| 430 | 10100 | 00000 | 43 |
| 431* | 00000 | 00000 | 431 |
| 432 | 43000 | 00000 | |
| 433* | 00000 | 00000 | 433 |
| 434 | 10010 | 00000 | 31 |
| 435 | 01100 | 00001 | |
| 436 | 20000 | 00000 | 109 |
| 437 | 00000 | 00110 | |
| 438 | 11000 | 00000 | 73 |
| 439* | 00000 | 00000 | 439 |
| 440 | 30101 | 00000 | |
| 441 | 02020 | 00000 | |
| 442 | 10000 | 11000 | |
| 443* | 00000 | 00000 | 443 |
| 444 | 21000 | 00000 | 37 |
| 445 | 00100 | 00000 | 89 |
| 446 | 10000 | 00000 | 223 |
| 447 | 01000 | 00000 | 149 |
| 448 | 60010 | 00000 | |
| 449* | 00000 | 00000 | 449 |
| 450 | 12200 | 00000 | |
| 451 | 00001 | 00000 | 41 |
| 452 | 20000 | 00000 | 113 |
| 453 | 01000 | 00000 | 151 |
| 454 | 10000 | 00000 | 227 |
| 455 | 00110 | 10000 | |
| 456 | 31000 | 00100 | |
| 457* | 00000 | 00000 | 457 |
| 458 | 10000 | 00000 | 229 |
| 459 | 03000 | 01000 | |
| 460 | 20100 | 00010 | |
| 461* | 00000 | 00000 | 461 |
| 462 | 11011 | 00000 | |
| 463* | 00000 | 00000 | 463 |
| 464 | 40000 | 00001 | |
| 465 | 01100 | 00000 | 31 |
| 466 | 10000 | 00000 | 233 |
| 467* | 00000 | 00000 | 467 |
| 468 | 22000 | 10000 | |
| 469 | 00010 | 00000 | 67 |
| 470 | 10100 | 00000 | 47 |
| 471 | 01000 | 00000 | 157 |
| 472 | 30000 | 00000 | 59 |
| 473 | 00001 | 00000 | 43 |
| 474 | 11000 | 00000 | 79 |
| 475 | 00200 | 00100 | |
| 476 | 20010 | 01000 | |
| 477 | 02000 | 00000 | 53 |
| 478 | 10000 | 00000 | 239 |
| 479* | 00000 | 00000 | 479 |
| 480 | 51100 | 00000 | |

| | 1<br>23571 | 11122<br>37939 | |
|---|---|---|---|
| 481 | 00000 | 10000 | 37 |
| 482 | 10000 | 00000 | 241 |
| 483 | 01010 | 00010 | |
| 484 | 20002 | 00000 | |
| 485 | 00100 | 00000 | 97 |
| 486 | 15000 | 00000 | |
| 487* | 00000 | 00000 | 487 |
| 488 | 30000 | 00000 | 61 |
| 489 | 01000 | 00000 | 163 |
| 490 | 10120 | 00000 | |
| 491* | 00000 | 00000 | 491 |
| 492 | 21000 | 00000 | 41 |
| 493 | 00000 | 01001 | |
| 494 | 10000 | 10100 | |
| 495 | 02101 | 00000 | |
| 496 | 40000 | 00000 | 31 |
| 497 | 00010 | 00000 | 71 |
| 498 | 11000 | 00000 | 83 |
| 499* | 00000 | 00000 | 499 |
| 500 | 20300 | 00000 | |
| 501 | 01000 | 00000 | 167 |
| 502 | 10000 | 00000 | 251 |
| 503* | 00000 | 00000 | 503 |
| 504 | 32010 | 00000 | |
| 505 | 00100 | 00000 | 101 |
| 506 | 10001 | 00010 | |
| 507 | 01000 | 20000 | |
| 508 | 20000 | 00000 | 127 |
| 509* | 00000 | 00000 | 509 |
| 510 | 11100 | 01000 | |
| 511 | 00010 | 00000 | 73 |
| 512 | 90000 | 00000 | |
| 513 | 03000 | 00100 | |
| 514 | 10000 | 00000 | 257 |
| 515 | 00100 | 00000 | 103 |
| 516 | 21000 | 00000 | 43 |
| 517 | 00001 | 00000 | 47 |
| 518 | 10010 | 00000 | 37 |
| 519 | 01000 | 00000 | 173 |
| 520 | 30100 | 10000 | |
| 521* | 00000 | 00000 | 521 |
| 522 | 12000 | 00001 | |
| 523* | 00000 | 00000 | 523 |
| 524 | 20000 | 00000 | 131 |
| 525 | 01210 | 00000 | |
| 526 | 10000 | 00000 | 263 |
| 527 | 00000 | 01000 | 31 |
| 528 | 41001 | 00000 | |
| 529 | 00000 | 00020 | |
| 530 | 10100 | 00000 | 53 |
| 531 | 02000 | 00000 | 59 |
| 532 | 20010 | 00100 | |
| 533 | 00000 | 10000 | 41 |
| 534 | 11000 | 00000 | 89 |
| 535 | 00100 | 00000 | 107 |
| 536 | 30000 | 00000 | 67 |
| 537 | 01000 | 00000 | 179 |
| 538 | 10000 | 00000 | 269 |
| 539 | 00021 | 00000 | |
| 540 | 23100 | 00000 | |

| | 1<br>23571 | 11122<br>37939 | |
|---|---|---|---|
| 541* | 00000 | 00000 | 541 |
| 542 | 10000 | 00000 | 271 |
| 543 | 01000 | 00000 | 181 |
| 544 | 50000 | 01000 | |
| 545 | 00100 | 00000 | 109 |
| 546 | 11010 | 10000 | |
| 547* | 00000 | 00000 | 547 |
| 548 | 20000 | 00000 | 137 |
| 549 | 02000 | 00000 | 61 |
| 550 | 10201 | 00000 | |
| 551 | 00000 | 00101 | |
| 552 | 31000 | 00010 | |
| 553 | 00010 | 00000 | 79 |
| 554 | 10000 | 00000 | 277 |
| 555 | 01100 | 00000 | 37 |
| 556 | 20000 | 00000 | 139 |
| 557* | 00000 | 00000 | 557 |
| 558 | 12000 | 00000 | 31 |
| 559 | 00000 | 10000 | 43 |
| 560 | 40110 | 00000 | |
| 561 | 01001 | 01000 | |
| 562 | 10000 | 00000 | 281 |
| 563* | 00000 | 00000 | 563 |
| 564 | 21000 | 00000 | 47 |
| 565 | 00100 | 00000 | 113 |
| 566 | 10000 | 00000 | 283 |
| 567 | 04010 | 00000 | |
| 568 | 30000 | 00000 | 71 |
| 569* | 00000 | 00000 | 569 |
| 570 | 11100 | 00100 | |
| 571* | 00000 | 00000 | 571 |
| 572 | 20001 | 10000 | |
| 573 | 01000 | 00000 | 191 |
| 574 | 10010 | 00000 | 41 |
| 575 | 00200 | 00010 | |
| 576 | 62000 | 00000 | |
| 577* | 00000 | 00000 | 577 |
| 578 | 10000 | 02000 | |
| 579 | 01000 | 00000 | 193 |
| 580 | 20100 | 00001 | |
| 581 | 00010 | 00000 | 83 |
| 582 | 11000 | 00000 | 97 |
| 583 | 00001 | 00000 | 53 |
| 584 | 30000 | 00000 | 73 |
| 585 | 02100 | 10000 | |
| 586 | 10000 | 00000 | 293 |
| 587* | 00000 | 00000 | 587 |
| 588 | 21020 | 00000 | |
| 589 | 00000 | 00100 | 31 |
| 590 | 10100 | 00000 | 59 |
| 591 | 01000 | 00000 | 197 |
| 592 | 40000 | 00000 | 37 |
| 593* | 00000 | 00000 | 593 |
| 594 | 13001 | 00000 | |
| 595 | 00110 | 01000 | |
| 596 | 20000 | 00000 | 149 |
| 597 | 01000 | 00000 | 199 |
| 598 | 10000 | 10010 | |
| 599* | 00000 | 00000 | 599 |
| 600 | 31200 | 00000 | |

| | 1<br>23571 | 11122<br>37939 | |
|---|---|---|---|
| 601* | 00000 | 00000 | 601 |
| 602 | 10010 | 00000 | 43 |
| 603 | 02000 | 00000 | 67 |
| 604 | 20000 | 00000 | 151 |
| 605 | 00102 | 00000 | |
| 606 | 11000 | 00000 | 101 |
| 607* | 00000 | 00000 | 607 |
| 608 | 50000 | 00100 | |
| 609 | 01010 | 00001 | |
| 610 | 10100 | 00000 | 61 |
| 611 | 00000 | 10000 | 47 |
| 612 | 22000 | 01000 | |
| 613* | 00000 | 00000 | 613 |
| 614 | 10000 | 00000 | 307 |
| 615 | 01100 | 00000 | 41 |
| 616 | 30011 | 00000 | |
| 617* | 00000 | 00000 | 617 |
| 618 | 11000 | 00000 | 103 |
| 619* | 00000 | 00000 | 619 |
| 620 | 20100 | 00000 | 31 |
| 621 | 03000 | 00010 | |
| 622 | 10000 | 00000 | 311 |
| 623 | 00010 | 00000 | 89 |
| 624 | 41000 | 10000 | |
| 625 | 00400 | 00000 | |
| 626 | 10000 | 00000 | 313 |
| 627 | 01001 | 00100 | |
| 628 | 20000 | 00000 | 157 |
| 629 | 00000 | 01000 | 37 |
| 630 | 12110 | 00000 | |
| 631* | 00000 | 00000 | 631 |
| 632 | 30000 | 00000 | 79 |
| 633 | 01000 | 00000 | 211 |
| 634 | 10000 | 00000 | 317 |
| 635 | 00100 | 00000 | 127 |
| 636 | 21000 | 00000 | 53 |
| 637 | 00020 | 10000 | |
| 638 | 10001 | 00001 | |
| 639 | 02000 | 00000 | 71 |
| 640 | 70100 | 00000 | |
| 641* | 00000 | 00000 | 641 |
| 642 | 11000 | 00000 | 107 |
| 643* | 00000 | 00000 | 643 |
| 644 | 20010 | 00010 | |
| 645 | 01100 | 00000 | 43 |
| 646 | 10000 | 01100 | |
| 647* | 00000 | 00000 | 647 |
| 648 | 34000 | 00000 | |
| 649 | 00001 | 00000 | 59 |
| 650 | 10200 | 10000 | |
| 651 | 01010 | 00000 | 31 |
| 652 | 20000 | 00000 | 163 |
| 653* | 00000 | 00000 | 653 |
| 654 | 11000 | 00000 | 109 |
| 655 | 00100 | 00000 | 131 |
| 656 | 40000 | 00000 | 41 |
| 657 | 02000 | 00000 | 73 |
| 658 | 10010 | 00000 | 47 |
| 659* | 00000 | 00000 | 659 |
| 660 | 21101 | 00000 | |

| | 1<br>23571 | 11122<br>37939 | |
|---|---|---|---|
| 661* | 00000 | 00000 | 661 |
| 662 | 10000 | 00000 | 331 |
| 663 | 01000 | 11000 | |
| 664 | 30000 | 00000 | 83 |
| 665 | 00110 | 00100 | |
| 666 | 12000 | 00000 | 37 |
| 667 | 00000 | 00011 | |
| 668 | 20000 | 00000 | 167 |
| 669 | 01000 | 00000 | 223 |
| 670 | 10100 | 00000 | 67 |
| 671 | 00001 | 00000 | 61 |
| 672 | 51010 | 00000 | |
| 673* | 00000 | 00000 | 673 |
| 674 | 10000 | 00000 | 337 |
| 675 | 03200 | 00000 | |
| 676 | 20000 | 20000 | |
| 677* | 00000 | 00000 | 677 |
| 678 | 11000 | 00000 | 113 |
| 679 | 00010 | 00000 | 97 |
| 680 | 30100 | 01000 | |
| 681 | 01000 | 00000 | 227 |
| 682 | 10001 | 00000 | 31 |
| 683* | 00000 | 00000 | 683 |
| 684 | 22000 | 00100 | |
| 685 | 00100 | 00000 | 137 |
| 686 | 10030 | 00000 | |
| 687 | 01000 | 00000 | 229 |
| 688 | 40000 | 00000 | 43 |
| 689 | 00000 | 10000 | 53 |
| 690 | 11100 | 00010 | |
| 691* | 00000 | 00000 | 691 |
| 692 | 20000 | 00000 | 173 |
| 693 | 02011 | 00000 | |
| 694 | 10000 | 00000 | 347 |
| 695 | 00100 | 00000 | 139 |
| 696 | 31000 | 00001 | |
| 697 | 00000 | 01000 | 41 |
| 698 | 10000 | 00000 | 349 |
| 699 | 01000 | 00000 | 233 |
| 700 | 20210 | 00000 | |
| 701* | 00000 | 00000 | 701 |
| 702 | 13000 | 10000 | |
| 703 | 00000 | 00100 | 37 |
| 704 | 60001 | 00000 | |
| 705 | 01100 | 00000 | 47 |
| 706 | 10000 | 00000 | 353 |
| 707 | 00010 | 00000 | 101 |
| 708 | 21000 | 00000 | 59 |
| 709* | 00000 | 00000 | 709 |
| 710 | 10100 | 00000 | 71 |
| 711 | 02000 | 00000 | 79 |
| 712 | 30000 | 00000 | 89 |
| 713 | 00000 | 00010 | 31 |
| 714 | 11010 | 01000 | |
| 715 | 00101 | 10000 | |
| 716 | 20000 | 00000 | 179 |
| 717 | 01000 | 00000 | 239 |
| 718 | 10000 | 00000 | 359 |
| 719* | 00000 | 00000 | 719 |
| 720 | 42100 | 00000 | |

| | 1 | 11122 | |
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| 723 | 01000 | 00000 | 241 |
| 724 | 20000 | 00000 | 181 |
| 725 | 00200 | 00001 | |
| 726 | 11002 | 00000 | |
| 727* | 00000 | 00000 | 727 |
| 728 | 30010 | 10000 | |
| 729 | 06000 | 00000 | |
| 730 | 10100 | 00000 | 73 |
| 731 | 00000 | 01000 | 43 |
| 732 | 21000 | 00000 | 61 |
| 733* | 00000 | 00000 | 733 |
| 734 | 10000 | 00000 | 367 |
| 735 | 01120 | 00000 | |
| 736 | 50000 | 00010 | |
| 737 | 00001 | 00000 | 67 |
| 738 | 12000 | 00000 | 41 |
| 739* | 00000 | 00000 | 739 |
| 740 | 20100 | 00000 | 37 |
| 741 | 01000 | 10100 | |
| 742 | 10010 | 00000 | 53 |
| 743* | 00000 | 00000 | 743 |
| 744 | 31000 | 00000 | 31 |
| 745 | 00100 | 00000 | 149 |
| 746 | 10000 | 00000 | 373 |
| 747 | 02000 | 00000 | 83 |
| 748 | 20001 | 01000 | |
| 749 | 00010 | 00000 | 107 |
| 750 | 11300 | 00000 | |
| 751* | 00000 | 00000 | 751 |
| 752 | 40000 | 00000 | 47 |
| 753 | 01000 | 00000 | 251 |
| 754 | 10000 | 10001 | |
| 755 | 00100 | 00000 | 151 |
| 756 | 23010 | 00000 | |
| 757* | 00000 | 00000 | 757 |
| 758 | 10000 | 00000 | 379 |
| 759 | 01001 | 00010 | |
| 760 | 30100 | 00100 | |
| 761* | 00000 | 00000 | 761 |
| 762 | 11000 | 00000 | 127 |
| 763 | 00010 | 00000 | 109 |
| 764 | 20000 | 00000 | 191 |
| 765 | 02100 | 01000 | |
| 766 | 10000 | 00000 | 383 |
| 767 | 00000 | 10000 | 59 |
| 768 | 81000 | 00000 | |
| 769* | 00000 | 00000 | 769 |
| 770 | 10111 | 00000 | |
| 771 | 01000 | 00000 | 257 |
| 772 | 20000 | 00000 | 193 |
| 773* | 00000 | 00000 | 773 |
| 774 | 12000 | 00000 | 43 |
| 775 | 00200 | 00000 | 31 |
| 776 | 30000 | 00000 | 97 |
| 777 | 01010 | 00000 | 37 |
| 778 | 10000 | 00000 | 389 |
| 779 | 00000 | 00100 | 41 |
| 780 | 21100 | 10000 | |
| 781 | 00001 | 00000 | 71 |
| 782 | 10000 | 01010 | |
| 783 | 03000 | 00001 | |
| 784 | 40020 | 00000 | |
| 785 | 00100 | 00000 | 157 |
| 786 | 11000 | 00000 | 131 |
| 787* | 00000 | 00000 | 787 |
| 788 | 20000 | 00000 | 197 |
| 789 | 01000 | 00000 | 263 |
| 790 | 10100 | 00000 | 79 |
| 791 | 00010 | 00000 | 113 |
| 792 | 32001 | 00000 | |
| 793 | 00000 | 10000 | 61 |
| 794 | 10000 | 00000 | 397 |
| 795 | 01100 | 00000 | 53 |
| 796 | 20000 | 00000 | 199 |
| 797* | 00000 | 00000 | 797 |
| 798 | 11010 | 00100 | |
| 799 | 00000 | 01000 | 47 |
| 800 | 50200 | 00000 | |
| 801 | 02000 | 00000 | 89 |
| 802 | 10000 | 00000 | 401 |
| 803 | 00001 | 00000 | 73 |
| 804 | 21000 | 00000 | 67 |
| 805 | 00110 | 00010 | |
| 806 | 10000 | 10000 | 31 |
| 807 | 01000 | 00000 | 269 |
| 808 | 30000 | 00000 | 101 |
| 809* | 00000 | 00000 | 809 |
| 810 | 14100 | 00000 | |
| 811* | 00000 | 00000 | 811 |
| 812 | 20010 | 00001 | |
| 813 | 01000 | 00000 | 271 |
| 814 | 10001 | 00000 | 37 |
| 815 | 00100 | 00000 | 163 |
| 816 | 41000 | 01000 | |
| 817 | 00000 | 00100 | 43 |
| 818 | 10000 | 00000 | 409 |
| 819 | 02010 | 10000 | |
| 820 | 20100 | 00000 | 41 |
| 821* | 00000 | 00000 | 821 |
| 822 | 11000 | 00000 | 137 |
| 823* | 00000 | 00000 | 823 |
| 824 | 30000 | 00000 | 103 |
| 825 | 01201 | 00000 | |
| 826 | 10010 | 00000 | 59 |
| 827* | 00000 | 00000 | 827 |
| 828 | 22000 | 00010 | |
| 829* | 00000 | 00000 | 829 |
| 830 | 10100 | 00000 | 83 |
| 831 | 01000 | 00000 | 277 |
| 832 | 60000 | 10000 | |
| 833 | 00020 | 01000 | |
| 834 | 11000 | 00000 | 139 |
| 835 | 00100 | 00000 | 167 |
| 836 | 20001 | 00100 | |
| 837 | 03000 | 00000 | 31 |
| 838 | 10000 | 00000 | 419 |
| 839* | 00000 | 00000 | 839 |
| 840 | 31110 | 00000 | |
| 841 | 00000 | 00002 | |
| 842 | 10000 | 00000 | 421 |
| 843 | 01000 | 00000 | 281 |
| 844 | 20000 | 00000 | 211 |
| 845 | 00100 | 20000 | |
| 846 | 12000 | 00000 | 47 |
| 847 | 00012 | 00000 | |
| 848 | 40000 | 00000 | 53 |
| 849 | 01000 | 00000 | 283 |
| 850 | 10200 | 01000 | |
| 851 | 00000 | 00010 | 37 |
| 852 | 21000 | 00000 | 71 |
| 853* | 00000 | 00000 | 853 |
| 854 | 10010 | 00000 | 61 |
| 855 | 02100 | 00100 | |
| 856 | 30000 | 00000 | 107 |
| 857* | 00000 | 00000 | 857 |
| 858 | 11001 | 10000 | |
| 859* | 00000 | 00000 | 859 |
| 860 | 20100 | 00000 | 43 |
| 861 | 01010 | 00000 | 41 |
| 862 | 10000 | 00000 | 431 |
| 863* | 00000 | 00000 | 863 |
| 864 | 53000 | 00000 | |
| 865 | 00100 | 00000 | 173 |
| 866 | 10000 | 00000 | 433 |
| 867 | 01000 | 02000 | |
| 868 | 20010 | 00000 | 31 |
| 869 | 00001 | 00000 | 79 |
| 870 | 11100 | 00001 | |
| 871 | 00000 | 10000 | 67 |
| 872 | 30000 | 00000 | 109 |
| 873 | 02000 | 00000 | 97 |
| 874 | 10000 | 00110 | |
| 875 | 00310 | 00000 | |
| 876 | 21000 | 00000 | 73 |
| 877* | 00000 | 00000 | 877 |
| 878 | 10000 | 00000 | 439 |
| 879 | 01000 | 00000 | 293 |
| 880 | 40101 | 00000 | |
| 881* | 00000 | 00000 | 881 |
| 882 | 12020 | 00000 | |
| 883* | 00000 | 00000 | 883 |
| 884 | 20000 | 11000 | |
| 885 | 01100 | 00000 | 59 |
| 886 | 10000 | 00000 | 443 |
| 887* | 00000 | 00000 | 887 |
| 888 | 31000 | 00000 | 37 |
| 889 | 00010 | 00000 | 127 |
| 890 | 10100 | 00000 | 89 |
| 891 | 04001 | 00000 | |
| 892 | 20000 | 00000 | 223 |
| 893 | 00000 | 00100 | 47 |
| 894 | 11000 | 00000 | 149 |
| 895 | 00100 | 00000 | 179 |
| 896 | 70010 | 00000 | |
| 897 | 01000 | 10010 | |
| 898 | 10000 | 00000 | 449 |
| 899 | 00000 | 00001 | 31 |
| 900 | 22200 | 00000 | |
| 901 | 00000 | 01000 | 53 |
| 902 | 10001 | 00000 | 41 |
| 903 | 01010 | 00000 | 43 |
| 904 | 30000 | 00000 | 113 |
| 905 | 00100 | 00000 | 181 |
| 906 | 11000 | 00000 | 151 |
| 907* | 00000 | 00000 | 907 |
| 908 | 20000 | 00000 | 227 |
| 909 | 02000 | 00000 | 101 |
| 910 | 10110 | 10000 | |
| 911* | 00000 | 00000 | 911 |
| 912 | 41000 | 00100 | |
| 913 | 00001 | 00000 | 83 |
| 914 | 10000 | 00000 | 457 |
| 915 | 01100 | 00000 | 61 |
| 916 | 20000 | 00000 | 229 |
| 917 | 00010 | 00000 | 131 |
| 918 | 13000 | 01000 | |
| 919* | 00000 | 00000 | 919 |
| 920 | 30100 | 00010 | |
| 921 | 01000 | 00000 | 307 |
| 922 | 10000 | 00000 | 461 |
| 923 | 00000 | 10000 | 71 |
| 924 | 21011 | 00000 | |
| 925 | 00200 | 00000 | 37 |
| 926 | 10000 | 00000 | 463 |
| 927 | 02000 | 00000 | 103 |
| 928 | 50000 | 00001 | |
| 929* | 00000 | 00000 | 929 |
| 930 | 11100 | 00000 | 31 |
| 931 | 00020 | 00100 | |
| 932 | 20000 | 00000 | 233 |
| 933 | 01000 | 00000 | 311 |
| 934 | 10000 | 00000 | 467 |
| 935 | 00101 | 01000 | |
| 936 | 32000 | 10000 | |
| 937* | 00000 | 00000 | 937 |
| 938 | 10010 | 00000 | 67 |
| 939 | 01000 | 00000 | 313 |
| 940 | 20100 | 00000 | 47 |
| 941* | 00000 | 00000 | 941 |
| 942 | 11000 | 00000 | 157 |
| 943 | 00000 | 00010 | 41 |
| 944 | 40000 | 00000 | 59 |
| 945 | 03110 | 00000 | |
| 946 | 10001 | 00000 | 43 |
| 947* | 00000 | 00000 | 947 |
| 948 | 21000 | 00000 | 79 |
| 949 | 00000 | 10000 | 73 |
| 950 | 10200 | 00100 | |
| 951 | 01000 | 00000 | 317 |
| 952 | 30010 | 01000 | |
| 953* | 00000 | 00000 | 953 |
| 954 | 12000 | 00000 | 53 |
| 955 | 00100 | 00000 | 191 |
| 956 | 20000 | 00000 | 239 |
| 957 | 01001 | 00001 | |
| 958 | 10000 | 00000 | 479 |
| 959 | 00010 | 00000 | 137 |
| 960 | 61100 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
| | 23571 | 37939 | |
| 1 | 00000 | 00000 | |
| 31* | 00000 | 00000 | 31 |
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| 43* | 00000 | 00000 | 43 |
| 47* | 00000 | 00000 | 47 |
| 53* | 00000 | 00000 | 53 |
| 59* | 00000 | 00000 | 59 |
| 61* | 00000 | 00000 | 61 |
| 67* | 00000 | 00000 | 67 |
| 71* | 00000 | 00000 | 71 |
| 73* | 00000 | 00000 | 73 |
| 79* | 00000 | 00000 | 79 |
| 83* | 00000 | 00000 | 83 |
| 89* | 00000 | 00000 | 89 |
| 97* | 00000 | 00000 | 97 |
| 101* | 00000 | 00000 | 101 |
| 103* | 00000 | 00000 | 103 |
| 107* | 00000 | 00000 | 107 |
| 109* | 00000 | 00000 | 109 |
| 113* | 00000 | 00000 | 113 |
| 127* | 00000 | 00000 | 127 |
| 131* | 00000 | 00000 | 131 |
| 137* | 00000 | 00000 | 137 |
| 139* | 00000 | 00000 | 139 |
| 149* | 00000 | 00000 | 149 |
| 151* | 00000 | 00000 | 151 |
| 157* | 00000 | 00000 | 157 |
| 163* | 00000 | 00000 | 163 |
| 167* | 00000 | 00000 | 167 |
| 173* | 00000 | 00000 | 173 |
| 179* | 00000 | 00000 | 179 |
| 181* | 00000 | 00000 | 181 |
| 191* | 00000 | 00000 | 191 |
| 193* | 00000 | 00000 | 193 |
| 197* | 00000 | 00000 | 197 |
| 199* | 00000 | 00000 | 199 |
| 211* | 00000 | 00000 | 211 |
| 223* | 00000 | 00000 | 223 |
| 227* | 00000 | 00000 | 227 |
| 229* | 00000 | 00000 | 229 |
| 233* | 00000 | 00000 | 233 |
| 239* | 00000 | 00000 | 239 |
| 241* | 00000 | 00000 | 241 |
| 251* | 00000 | 00000 | 251 |
| 257* | 00000 | 00000 | 257 |
| 263* | 00000 | 00000 | 263 |
| 269* | 00000 | 00000 | 269 |
| 271* | 00000 | 00000 | 271 |
| 277* | 00000 | 00000 | 277 |
| 281* | 00000 | 00000 | 281 |
| 283* | 00000 | 00000 | 283 |
| 293* | 00000 | 00000 | 293 |
| 307* | 00000 | 00000 | 307 |
| 311* | 00000 | 00000 | 311 |
| 313* | 00000 | 00000 | 313 |
| 317* | 00000 | 00000 | 317 |
| 331* | 00000 | 00000 | 331 |
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| 347* | 00000 | 00000 | 347 |
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| 353* | 00000 | 00000 | 353 |
| 359* | 00000 | 00000 | 359 |
| 367* | 00000 | 00000 | 367 |
| 373* | 00000 | 00000 | 373 |
| 379* | 00000 | 00000 | 379 |
| 383* | 00000 | 00000 | 383 |
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| 397* | 00000 | 00000 | 397 |
| 401* | 00000 | 00000 | 401 |
| 409* | 00000 | 00000 | 409 |
| 419* | 00000 | 00000 | 419 |
| 421* | 00000 | 00000 | 421 |
| 431* | 00000 | 00000 | 431 |
| 433* | 00000 | 00000 | 433 |
| 439* | 00000 | 00000 | 439 |
| 443* | 00000 | 00000 | 443 |
| 449* | 00000 | 00000 | 449 |
| 457* | 00000 | 00000 | 457 |
| 461* | 00000 | 00000 | 461 |
| 463* | 00000 | 00000 | 463 |
| 467* | 00000 | 00000 | 467 |
| 479* | 00000 | 00000 | 479 |
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| 499* | 00000 | 00000 | 499 |
| 503* | 00000 | 00000 | 503 |
| 509* | 00000 | 00000 | 509 |
| 521* | 00000 | 00000 | 521 |
| 523* | 00000 | 00000 | 523 |
| 541* | 00000 | 00000 | 541 |
| 547* | 00000 | 00000 | 547 |
| 557* | 00000 | 00000 | 557 |
| 563* | 00000 | 00000 | 563 |
| 569* | 00000 | 00000 | 569 |
| 571* | 00000 | 00000 | 571 |
| 577* | 00000 | 00000 | 577 |
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| 593* | 00000 | 00000 | 593 |
| 599* | 00000 | 00000 | 599 |
| 601* | 00000 | 00000 | 601 |
| 607* | 00000 | 00000 | 607 |
| 613* | 00000 | 00000 | 613 |
| 617* | 00000 | 00000 | 617 |
| 619* | 00000 | 00000 | 619 |
| 631* | 00000 | 00000 | 631 |
| 641* | 00000 | 00000 | 641 |
| 643* | 00000 | 00000 | 643 |
| 647* | 00000 | 00000 | 647 |
| 653* | 00000 | 00000 | 653 |
| 659* | 00000 | 00000 | 659 |
| 661* | 00000 | 00000 | 661 |
| 673* | 00000 | 00000 | 673 |
| 677* | 00000 | 00000 | 677 |
| 683* | 00000 | 00000 | 683 |
| 691* | 00000 | 00000 | 691 |
| 701* | 00000 | 00000 | 701 |
| 709* | 00000 | 00000 | 709 |
| 719* | 00000 | 00000 | 719 |
| 727* | 00000 | 00000 | 727 |
| 733* | 00000 | 00000 | 733 |
| 739* | 00000 | 00000 | 739 |
| 743* | 00000 | 00000 | 743 |
| 751* | 00000 | 00000 | 751 |
| 757* | 00000 | 00000 | 757 |
| 761* | 00000 | 00000 | 761 |
| 769* | 00000 | 00000 | 769 |
| 773* | 00000 | 00000 | 773 |
| 787* | 00000 | 00000 | 787 |
| 797* | 00000 | 00000 | 797 |
| 809* | 00000 | 00000 | 809 |
| 811* | 00000 | 00000 | 811 |
| 821* | 00000 | 00000 | 821 |
| 823* | 00000 | 00000 | 823 |
| 827* | 00000 | 00000 | 827 |
| 829* | 00000 | 00000 | 829 |
| 839* | 00000 | 00000 | 839 |
| 853* | 00000 | 00000 | 853 |
| 857* | 00000 | 00000 | 857 |
| 859* | 00000 | 00000 | 859 |
| 863* | 00000 | 00000 | 863 |
| 877* | 00000 | 00000 | 877 |
| 881* | 00000 | 00000 | 881 |
| 883* | 00000 | 00000 | 883 |
| 887* | 00000 | 00000 | 887 |
| 907* | 00000 | 00000 | 907 |
| 911* | 00000 | 00000 | 911 |
| 919* | 00000 | 00000 | 919 |
| 929* | 00000 | 00000 | 929 |
| 937* | 00000 | 00000 | 937 |
| 941* | 00000 | 00000 | 941 |
| 947* | 00000 | 00000 | 947 |
| 953* | 00000 | 00000 | 953 |
| 29* | 00000 | 00001 | |
| 899 | 00000 | 00001 | 31 |
| 841 | 00000 | 00002 | |
| 23* | 00000 | 00010 | |
| 851 | 00000 | 00010 | 37 |
| 943 | 00000 | 00010 | 41 |
| 713 | 00000 | 00010 | 31 |
| 667 | 00000 | 00011 | |
| 529 | 00000 | 00020 | |
| 19* | 00000 | 00100 | |
| 703 | 00000 | 00100 | 37 |
| 779 | 00000 | 00100 | 41 |
| 817 | 00000 | 00100 | 43 |
| 893 | 00000 | 00100 | 47 |
| 589 | 00000 | 00100 | 31 |
| 551 | 00000 | 00101 | |
| 437 | 00000 | 00110 | |
| 361 | 00000 | 00200 | |
| 17* | 00000 | 01000 | |
| 629 | 00000 | 01000 | 37 |
| 697 | 00000 | 01000 | 41 |
| 731 | 00000 | 01000 | 43 |
| 799 | 00000 | 01000 | 47 |
| 901 | 00000 | 01000 | 53 |
| 527 | 00000 | 01000 | 31 |
| 493 | 00000 | 01001 | |
| 391 | 00000 | 01010 | |
| 323 | 00000 | 01100 | |
| 289 | 00000 | 02000 | |
| 13* | 00000 | 10000 | |
| 481 | 00000 | 10000 | 37 |
| 533 | 00000 | 10000 | 41 |
| 559 | 00000 | 10000 | 43 |
| 611 | 00000 | 10000 | 47 |
| 689 | 00000 | 10000 | 53 |
| 767 | 00000 | 10000 | 59 |
| 793 | 00000 | 10000 | 61 |
| 871 | 00000 | 10000 | 67 |
| 923 | 00000 | 10000 | 71 |
| 949 | 00000 | 10000 | 73 |
| 403 | 00000 | 10000 | 31 |
| 377 | 00000 | 10001 | |
| 299 | 00000 | 10010 | |
| 247 | 00000 | 10100 | |
| 221 | 00000 | 11000 | |
| 169 | 00000 | 20000 | |
| 11* | 00001 | 00000 | |
| 407 | 00001 | 00000 | 37 |
| 451 | 00001 | 00000 | 41 |
| 473 | 00001 | 00000 | 43 |
| 517 | 00001 | 00000 | 47 |
| 583 | 00001 | 00000 | 53 |
| 649 | 00001 | 00000 | 59 |
| 671 | 00001 | 00000 | 61 |
| 737 | 00001 | 00000 | 67 |
| 781 | 00001 | 00000 | 71 |
| 803 | 00001 | 00000 | 73 |
| 869 | 00001 | 00000 | 79 |
| 913 | 00001 | 00000 | 83 |
| 341 | 00001 | 00000 | 31 |
| 319 | 00001 | 00001 | |
| 253 | 00001 | 00010 | |
| 209 | 00001 | 00100 | |
| 187 | 00001 | 01000 | |
| 143 | 00001 | 10000 | |
| 121 | 00002 | 00000 | |
| 7* | 00010 | 00000 | |
| 259 | 00010 | 00000 | 37 |
| 287 | 00010 | 00000 | 41 |
| 301 | 00010 | 00000 | 43 |
| 329 | 00010 | 00000 | 47 |
| 371 | 00010 | 00000 | 53 |
| 413 | 00010 | 00000 | 59 |
| 427 | 00010 | 00000 | 61 |
| 469 | 00010 | 00000 | 67 |
| 497 | 00010 | 00000 | 71 |
| 511 | 00010 | 00000 | 73 |
| 553 | 00010 | 00000 | 79 |
| 581 | 00010 | 00000 | 83 |
| 623 | 00010 | 00000 | 89 |
| 679 | 00010 | 00000 | 97 |
| 707 | 00010 | 00000 | 101 |
| 721 | 00010 | 00000 | 103 |
| 749 | 00010 | 00000 | 107 |
| 763 | 00010 | 00000 | 109 |
| 791 | 00010 | 00000 | 113 |
| 889 | 00010 | 00000 | 127 |

| | 1<br>23571 | 11122<br>37939 | |
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| 917 | 00010 | 00000 | 131 |
| 959 | 00010 | 00000 | 137 |
| 217 | 00010 | 00000 | 31 |
| 203 | 00010 | 00001 | |
| 161 | 00010 | 00010 | |
| 133 | 00010 | 00100 | |
| 119 | 00010 | 01000 | |
| 91 | 00010 | 10000 | |
| 77 | 00011 | 00000 | |
| 847 | 00012 | 00000 | |
| 49 | 00020 | 00000 | |
| 931 | 00020 | 00100 | |
| 833 | 00020 | 01000 | |
| 637 | 00020 | 10000 | |
| 539 | 00021 | 00000 | |
| 343 | 00030 | 00000 | |
| 5* | 00100 | 00000 | |
| 185 | 00100 | 00000 | 37 |
| 205 | 00100 | 00000 | 41 |
| 215 | 00100 | 00000 | 43 |
| 235 | 00100 | 00000 | 47 |
| 265 | 00100 | 00000 | 53 |
| 295 | 00100 | 00000 | 59 |
| 305 | 00100 | 00000 | 61 |
| 335 | 00100 | 00000 | 67 |
| 355 | 00100 | 00000 | 71 |
| 365 | 00100 | 00000 | 73 |
| 395 | 00100 | 00000 | 79 |
| 415 | 00100 | 00000 | 83 |
| 445 | 00100 | 00000 | 89 |
| 485 | 00100 | 00000 | 97 |
| 505 | 00100 | 00000 | 101 |
| 515 | 00100 | 00000 | 103 |
| 535 | 00100 | 00000 | 107 |
| 545 | 00100 | 00000 | 109 |
| 565 | 00100 | 00000 | 113 |
| 635 | 00100 | 00000 | 127 |
| 655 | 00100 | 00000 | 131 |
| 685 | 00100 | 00000 | 137 |
| 695 | 00100 | 00000 | 139 |
| 745 | 00100 | 00000 | 149 |
| 755 | 00100 | 00000 | 151 |
| 785 | 00100 | 00000 | 157 |
| 815 | 00100 | 00000 | 163 |
| 835 | 00100 | 00000 | 167 |
| 865 | 00100 | 00000 | 173 |
| 895 | 00100 | 00000 | 179 |
| 905 | 00100 | 00000 | 181 |
| 955 | 00100 | 00000 | 191 |
| 155 | 00100 | 00000 | 31 |
| 145 | 00100 | 00001 | |
| 115 | 00100 | 00010 | |
| 95 | 00100 | 00100 | |
| 85 | 00100 | 01000 | |
| 65 | 00100 | 10000 | |
| 845 | 00100 | 20000 | |
| 55 | 00101 | 00000 | |
| 935 | 00101 | 01000 | |
| 715 | 00101 | 10000 | |
| 605 | 00102 | 00000 | |

| | 1<br>23571 | 11122<br>37939 | |
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| 35 | 00110 | 00000 | |
| 805 | 00110 | 00010 | |
| 665 | 00110 | 00100 | |
| 595 | 00110 | 01000 | |
| 455 | 00110 | 10000 | |
| 385 | 00111 | 00000 | |
| 245 | 00120 | 00000 | |
| 25 | 00200 | 00000 | |
| 925 | 00200 | 00000 | 37 |
| 775 | 00200 | 00000 | 31 |
| 725 | 00200 | 00001 | |
| 575 | 00200 | 00010 | |
| 475 | 00200 | 00100 | |
| 425 | 00200 | 01000 | |
| 325 | 00200 | 10000 | |
| 275 | 00201 | 00000 | |
| 175 | 00210 | 00000 | |
| 125 | 00300 | 00000 | |
| 875 | 00310 | 00000 | |
| 625 | 00400 | 00000 | |
| 3* | 01000 | 00000 | |
| 111 | 01000 | 00000 | 37 |
| 123 | 01000 | 00000 | 41 |
| 129 | 01000 | 00000 | 43 |
| 141 | 01000 | 00000 | 47 |
| 159 | 01000 | 00000 | 53 |
| 177 | 01000 | 00000 | 59 |
| 183 | 01000 | 00000 | 61 |
| 201 | 01000 | 00000 | 67 |
| 213 | 01000 | 00000 | 71 |
| 219 | 01000 | 00000 | 73 |
| 237 | 01000 | 00000 | 79 |
| 249 | 01000 | 00000 | 83 |
| 267 | 01000 | 00000 | 89 |
| 291 | 01000 | 00000 | 97 |
| 303 | 01000 | 00000 | 101 |
| 309 | 01000 | 00000 | 103 |
| 321 | 01000 | 00000 | 107 |
| 327 | 01000 | 00000 | 109 |
| 339 | 01000 | 00000 | 113 |
| 381 | 01000 | 00000 | 127 |
| 393 | 01000 | 00000 | 131 |
| 411 | 01000 | 00000 | 137 |
| 417 | 01000 | 00000 | 139 |
| 447 | 01000 | 00000 | 149 |
| 453 | 01000 | 00000 | 151 |
| 471 | 01000 | 00000 | 157 |
| 489 | 01000 | 00000 | 163 |
| 501 | 01000 | 00000 | 167 |
| 519 | 01000 | 00000 | 173 |
| 537 | 01000 | 00000 | 179 |
| 543 | 01000 | 00000 | 181 |
| 573 | 01000 | 00000 | 191 |
| 579 | 01000 | 00000 | 193 |
| 591 | 01000 | 00000 | 197 |
| 597 | 01000 | 00000 | 199 |
| 633 | 01000 | 00000 | 211 |
| 669 | 01000 | 00000 | 223 |
| 681 | 01000 | 00000 | 227 |
| 687 | 01000 | 00000 | 229 |

| | 1<br>23571 | 11122<br>37939 | |
|---|---|---|---|
| 699 | 01000 | 00000 | 233 |
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| 723 | 01000 | 00000 | 241 |
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| 771 | 01000 | 00000 | 257 |
| 789 | 01000 | 00000 | 263 |
| 807 | 01000 | 00000 | 269 |
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| 831 | 01000 | 00000 | 277 |
| 843 | 01000 | 00000 | 281 |
| 849 | 01000 | 00000 | 283 |
| 879 | 01000 | 00000 | 293 |
| 921 | 01000 | 00000 | 307 |
| 933 | 01000 | 00000 | 311 |
| 939 | 01000 | 00000 | 313 |
| 951 | 01000 | 00000 | 317 |
| 93 | 01000 | 00000 | 31 |
| 87 | 01000 | 00001 | |
| 69 | 01000 | 00010 | |
| 57 | 01000 | 00100 | |
| 51 | 01000 | 01000 | |
| 867 | 01000 | 02000 | |
| 39 | 01000 | 10000 | |
| 897 | 01000 | 10010 | |
| 741 | 01000 | 10100 | |
| 663 | 01000 | 11000 | |
| 507 | 01000 | 20000 | |
| 33 | 01001 | 00000 | |
| 957 | 01001 | 00001 | |
| 759 | 01001 | 00010 | |
| 627 | 01001 | 00100 | |
| 561 | 01001 | 01000 | |
| 429 | 01001 | 10000 | |
| 363 | 01002 | 00000 | |
| 21 | 01010 | 00000 | |
| 777 | 01010 | 00000 | 37 |
| 861 | 01010 | 00000 | 41 |
| 903 | 01010 | 00000 | 43 |
| 651 | 01010 | 00000 | 31 |
| 609 | 01010 | 00001 | |
| 483 | 01010 | 00010 | |
| 399 | 01010 | 00100 | |
| 357 | 01010 | 01000 | |
| 273 | 01010 | 10000 | |
| 231 | 01011 | 00000 | |
| 147 | 01020 | 00000 | |
| 15 | 01100 | 00000 | |
| 555 | 01100 | 00000 | 37 |
| 615 | 01100 | 00000 | 41 |
| 645 | 01100 | 00000 | 43 |
| 705 | 01100 | 00000 | 47 |
| 795 | 01100 | 00000 | 53 |
| 885 | 01100 | 00000 | 59 |
| 915 | 01100 | 00000 | 61 |
| 465 | 01100 | 00000 | 31 |
| 435 | 01100 | 00001 | |
| 345 | 01100 | 00010 | |
| 285 | 01100 | 00100 | |
| 255 | 01100 | 01000 | |
| 195 | 01100 | 10000 | |

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| 735 | 01120 | 00000 | |
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| 525 | 01210 | 00000 | |
| 375 | 01300 | 00000 | |
| 9 | 02000 | 00000 | |
| 333 | 02000 | 00000 | 37 |
| 369 | 02000 | 00000 | 41 |
| 387 | 02000 | 00000 | 43 |
| 423 | 02000 | 00000 | 47 |
| 477 | 02000 | 00000 | 53 |
| 531 | 02000 | 00000 | 59 |
| 549 | 02000 | 00000 | 61 |
| 603 | 02000 | 00000 | 67 |
| 639 | 02000 | 00000 | 71 |
| 657 | 02000 | 00000 | 73 |
| 711 | 02000 | 00000 | 79 |
| 747 | 02000 | 00000 | 83 |
| 801 | 02000 | 00000 | 89 |
| 873 | 02000 | 00000 | 97 |
| 909 | 02000 | 00000 | 101 |
| 927 | 02000 | 00000 | 103 |
| 279 | 02000 | 00000 | 31 |
| 261 | 02000 | 00001 | |
| 207 | 02000 | 00010 | |
| 171 | 02000 | 00100 | |
| 153 | 02000 | 01000 | |
| 117 | 02000 | 10000 | |
| 99 | 02001 | 00000 | |
| 63 | 02010 | 00000 | |
| 819 | 02010 | 10000 | |
| 693 | 02011 | 00000 | |
| 441 | 02020 | 00000 | |
| 45 | 02100 | 00000 | |
| 855 | 02100 | 00100 | |
| 765 | 02100 | 01000 | |
| 585 | 02100 | 10000 | |
| 495 | 02101 | 00000 | |
| 315 | 02110 | 00000 | |
| 225 | 02200 | 00000 | |
| 27 | 03000 | 00000 | |
| 837 | 03000 | 00000 | 31 |
| 783 | 03000 | 00001 | |
| 621 | 03000 | 00010 | |
| 513 | 03000 | 00100 | |
| 459 | 03000 | 01000 | |
| 351 | 03000 | 10000 | |
| 297 | 03001 | 00000 | |
| 189 | 03010 | 00000 | |
| 135 | 03100 | 00000 | |
| 945 | 03110 | 00000 | |
| 675 | 03200 | 00000 | |
| 81 | 04000 | 00000 | |
| 891 | 04001 | 00000 | |
| 567 | 04010 | 00000 | |
| 405 | 04100 | 00000 | |
| 243 | 05000 | 00000 | |
| 729 | 06000 | 00000 | |

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| 106 | 10000 | 00000 | 53 |
| 118 | 10000 | 00000 | 59 |
| 122 | 10000 | 00000 | 61 |
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| 142 | 10000 | 00000 | 71 |
| 146 | 10000 | 00000 | 73 |
| 158 | 10000 | 00000 | 79 |
| 166 | 10000 | 00000 | 83 |
| 178 | 10000 | 00000 | 89 |
| 194 | 10000 | 00000 | 97 |
| 202 | 10000 | 00000 | 101 |
| 206 | 10000 | 00000 | 103 |
| 214 | 10000 | 00000 | 107 |
| 218 | 10000 | 00000 | 109 |
| 226 | 10000 | 00000 | 113 |
| 254 | 10000 | 00000 | 127 |
| 262 | 10000 | 00000 | 131 |
| 274 | 10000 | 00000 | 137 |
| 278 | 10000 | 00000 | 139 |
| 298 | 10000 | 00000 | 149 |
| 302 | 10000 | 00000 | 151 |
| 314 | 10000 | 00000 | 157 |
| 326 | 10000 | 00000 | 163 |
| 334 | 10000 | 00000 | 167 |
| 346 | 10000 | 00000 | 173 |
| 358 | 10000 | 00000 | 179 |
| 362 | 10000 | 00000 | 181 |
| 382 | 10000 | 00000 | 191 |
| 386 | 10000 | 00000 | 193 |
| 394 | 10000 | 00000 | 197 |
| 398 | 10000 | 00000 | 199 |
| 422 | 10000 | 00000 | 211 |
| 446 | 10000 | 00000 | 223 |
| 454 | 10000 | 00000 | 227 |
| 458 | 10000 | 00000 | 229 |
| 466 | 10000 | 00000 | 233 |
| 478 | 10000 | 00000 | 239 |
| 482 | 10000 | 00000 | 241 |
| 502 | 10000 | 00000 | 251 |
| 514 | 10000 | 00000 | 257 |
| 526 | 10000 | 00000 | 263 |
| 538 | 10000 | 00000 | 269 |
| 542 | 10000 | 00000 | 271 |
| 554 | 10000 | 00000 | 277 |
| 562 | 10000 | 00000 | 281 |
| 566 | 10000 | 00000 | 283 |
| 586 | 10000 | 00000 | 293 |
| 614 | 10000 | 00000 | 307 |
| 622 | 10000 | 00000 | 311 |
| 626 | 10000 | 00000 | 313 |
| 634 | 10000 | 00000 | 317 |
| 662 | 10000 | 00000 | 331 |
| 674 | 10000 | 00000 | 337 |
| 694 | 10000 | 00000 | 347 |
| 698 | 10000 | 00000 | 349 |
| 706 | 10000 | 00000 | 353 |
| 718 | 10000 | 00000 | 359 |
| 734 | 10000 | 00000 | 367 |
| 746 | 10000 | 00000 | 373 |
| 758 | 10000 | 00000 | 379 |
| 766 | 10000 | 00000 | 383 |
| 778 | 10000 | 00000 | 389 |
| 794 | 10000 | 00000 | 397 |
| 802 | 10000 | 00000 | 401 |
| 818 | 10000 | 00000 | 409 |
| 838 | 10000 | 00000 | 419 |
| 842 | 10000 | 00000 | 421 |
| 862 | 10000 | 00000 | 431 |
| 866 | 10000 | 00000 | 433 |
| 878 | 10000 | 00000 | 439 |
| 886 | 10000 | 00000 | 443 |
| 898 | 10000 | 00000 | 449 |
| 914 | 10000 | 00000 | 457 |
| 922 | 10000 | 00000 | 461 |
| 926 | 10000 | 00000 | 463 |
| 934 | 10000 | 00000 | 467 |
| 958 | 10000 | 00000 | 479 |
| 62 | 10000 | 00000 | 31 |
| 58 | 10000 | 00001 | |
| 46 | 10000 | 00010 | |
| 38 | 10000 | 00100 | |
| 874 | 10000 | 00110 | |
| 722 | 10000 | 00200 | |
| 34 | 10000 | 01000 | |
| 782 | 10000 | 01010 | |
| 646 | 10000 | 01100 | |
| 578 | 10000 | 02000 | |
| 26 | 10000 | 10000 | |
| 806 | 10000 | 10000 | 31 |
| 754 | 10000 | 10001 | |
| 598 | 10000 | 10010 | |
| 494 | 10000 | 10100 | |
| 442 | 10000 | 11000 | |
| 338 | 10000 | 20000 | |
| 22 | 10001 | 00000 | |
| 814 | 10001 | 00000 | 37 |
| 902 | 10001 | 00000 | 41 |
| 946 | 10001 | 00000 | 43 |
| 682 | 10001 | 00000 | 31 |
| 638 | 10001 | 00001 | |
| 506 | 10001 | 00010 | |
| 418 | 10001 | 00100 | |
| 374 | 10001 | 01000 | |
| 286 | 10001 | 10000 | |
| 242 | 10002 | 00000 | |
| 14 | 10010 | 00000 | |
| 518 | 10010 | 00000 | 37 |
| 574 | 10010 | 00000 | 41 |
| 602 | 10010 | 00000 | 43 |
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| 742 | 10010 | 00000 | 53 |
| 826 | 10010 | 00000 | 59 |
| 854 | 10010 | 00000 | 61 |
| 938 | 10010 | 00000 | 67 |
| 434 | 10010 | 00000 | 31 |
| 406 | 10010 | 00001 | |
| 322 | 10010 | 00010 | |
| 266 | 10010 | 00100 | |
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| 686 | 10030 | 00000 | |
| 10 | 10100 | 00000 | |
| 370 | 10100 | 00000 | 37 |
| 410 | 10100 | 00000 | 41 |
| 430 | 10100 | 00000 | 43 |
| 470 | 10100 | 00000 | 47 |
| 530 | 10100 | 00000 | 53 |
| 590 | 10100 | 00000 | 59 |
| 610 | 10100 | 00000 | 61 |
| 670 | 10100 | 00000 | 67 |
| 710 | 10100 | 00000 | 71 |
| 730 | 10100 | 00000 | 73 |
| 790 | 10100 | 00000 | 79 |
| 830 | 10100 | 00000 | 83 |
| 890 | 10100 | 00000 | 89 |
| 310 | 10100 | 00000 | 31 |
| 290 | 10100 | 00001 | |
| 230 | 10100 | 00010 | |
| 190 | 10100 | 00100 | |
| 170 | 10100 | 01000 | |
| 130 | 10100 | 10000 | |
| 110 | 10101 | 00000 | |
| 70 | 10110 | 00000 | |
| 910 | 10110 | 10000 | |
| 770 | 10111 | 00000 | |
| 490 | 10120 | 00000 | |
| 50 | 10200 | 00000 | |
| 950 | 10200 | 00100 | |
| 850 | 10200 | 01000 | |
| 650 | 10200 | 10000 | |
| 550 | 10201 | 00000 | |
| 350 | 10210 | 00000 | |
| 250 | 10300 | 00000 | |
| 6 | 11000 | 00000 | |
| 222 | 11000 | 00000 | 37 |
| 246 | 11000 | 00000 | 41 |
| 258 | 11000 | 00000 | 43 |
| 282 | 11000 | 00000 | 47 |
| 318 | 11000 | 00000 | 53 |
| 354 | 11000 | 00000 | 59 |
| 366 | 11000 | 00000 | 61 |
| 402 | 11000 | 00000 | 67 |
| 426 | 11000 | 00000 | 71 |
| 438 | 11000 | 00000 | 73 |
| 474 | 11000 | 00000 | 79 |
| 498 | 11000 | 00000 | 83 |
| 534 | 11000 | 00000 | 89 |
| 582 | 11000 | 00000 | 97 |
| 606 | 11000 | 00000 | 101 |
| 618 | 11000 | 00000 | 103 |
| 642 | 11000 | 00000 | 107 |
| 654 | 11000 | 00000 | 109 |
| 678 | 11000 | 00000 | 113 |
| 762 | 11000 | 00000 | 127 |
| 786 | 11000 | 00000 | 131 |
| 822 | 11000 | 00000 | 137 |
| 834 | 11000 | 00000 | 139 |
| 894 | 11000 | 00000 | 149 |
| 906 | 11000 | 00000 | 151 |
| 942 | 11000 | 00000 | 157 |
| 186 | 11000 | 00000 | 31 |
| 174 | 11000 | 00001 | |
| 138 | 11000 | 00010 | |
| 114 | 11000 | 00100 | |
| 102 | 11000 | 01000 | |
| 78 | 11000 | 10000 | |
| 66 | 11001 | 00000 | |
| 858 | 11001 | 10000 | |
| 726 | 11002 | 00000 | |
| 42 | 11010 | 00000 | |
| 798 | 11010 | 00100 | |
| 714 | 11010 | 01000 | |
| 546 | 11010 | 10000 | |
| 462 | 11011 | 00000 | |
| 294 | 11020 | 00000 | |
| 30 | 11100 | 00000 | |
| 930 | 11100 | 00000 | 31 |
| 870 | 11100 | 00001 | |
| 690 | 11100 | 00010 | |
| 570 | 11100 | 00100 | |
| 510 | 11100 | 01000 | |
| 390 | 11100 | 10000 | |
| 330 | 11101 | 00000 | |
| 210 | 11110 | 00000 | |
| 150 | 11200 | 00000 | |
| 750 | 11300 | 00000 | |
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| 666 | 12000 | 00000 | 37 |
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| 774 | 12000 | 00000 | 43 |
| 846 | 12000 | 00000 | 47 |
| 954 | 12000 | 00000 | 53 |
| 558 | 12000 | 00000 | 31 |
| 522 | 12000 | 00001 | |
| 414 | 12000 | 00010 | |
| 342 | 12000 | 00100 | |
| 306 | 12000 | 01000 | |
| 234 | 12000 | 10000 | |
| 198 | 12001 | 00000 | |
| 126 | 12010 | 00000 | |
| 882 | 12020 | 00000 | |
| 90 | 12100 | 00000 | |
| 630 | 12110 | 00000 | |
| 450 | 12200 | 00000 | |
| 54 | 13000 | 00000 | |
| 918 | 13000 | 01000 | |
| 702 | 13000 | 10000 | |
| 594 | 13001 | 00000 | |
| 378 | 13010 | 00000 | |
| 270 | 13100 | 00000 | |
| 162 | 14000 | 00000 | |
| 810 | 14100 | 00000 | |
| 486 | 15000 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
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| 164 | 20000 | 00000 | 41 |
| 172 | 20000 | 00000 | 43 |
| 188 | 20000 | 00000 | 47 |
| 212 | 20000 | 00000 | 53 |
| 236 | 20000 | 00000 | 59 |
| 244 | 20000 | 00000 | 61 |
| 268 | 20000 | 00000 | 67 |
| 284 | 20000 | 00000 | 71 |
| 292 | 20000 | 00000 | 73 |
| 316 | 20000 | 00000 | 79 |
| 332 | 20000 | 00000 | 83 |
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| 388 | 20000 | 00000 | 97 |
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| 412 | 20000 | 00000 | 103 |
| 428 | 20000 | 00000 | 107 |
| 436 | 20000 | 00000 | 109 |
| 452 | 20000 | 00000 | 113 |
| 508 | 20000 | 00000 | 127 |
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| 556 | 20000 | 00000 | 139 |
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| 724 | 20000 | 00000 | 181 |
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| 772 | 20000 | 00000 | 193 |
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| 796 | 20000 | 00000 | 199 |
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| 892 | 20000 | 00000 | 223 |
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| 956 | 20000 | 00000 | 239 |
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| 92 | 20000 | 00010 | |
| 76 | 20000 | 00100 | |
| 68 | 20000 | 01000 | |
| 52 | 20000 | 10000 | |
| 884 | 20000 | 11000 | |
| 676 | 20000 | 20000 | |
| 44 | 20001 | 00000 | |
| 836 | 20001 | 00100 | |
| 748 | 20001 | 01000 | |
| 572 | 20001 | 10000 | |
| 484 | 20002 | 00000 | |
| 28 | 20010 | 00000 | |
| 868 | 20010 | 00000 | 31 |
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| 644 | 20010 | 00010 | |
| 532 | 20010 | 00100 | |

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| 364 | 20010 | 10000 | |
| 308 | 20011 | 00000 | |
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| 20 | 20100 | 00000 | |
| 740 | 20100 | 00000 | 37 |
| 820 | 20100 | 00000 | 41 |
| 860 | 20100 | 00000 | 43 |
| 940 | 20100 | 00000 | 47 |
| 620 | 20100 | 00000 | 31 |
| 580 | 20100 | 00001 | |
| 460 | 20100 | 00010 | |
| 380 | 20100 | 00100 | |
| 340 | 20100 | 01000 | |
| 260 | 20100 | 10000 | |
| 220 | 20101 | 00000 | |
| 140 | 20110 | 00000 | |
| 100 | 20200 | 00000 | |
| 700 | 20210 | 00000 | |
| 500 | 20300 | 00000 | |
| 12 | 21000 | 00000 | |
| 444 | 21000 | 00000 | 37 |
| 492 | 21000 | 00000 | 41 |
| 516 | 21000 | 00000 | 43 |
| 564 | 21000 | 00000 | 47 |
| 636 | 21000 | 00000 | 53 |
| 708 | 21000 | 00000 | 59 |
| 732 | 21000 | 00000 | 61 |
| 804 | 21000 | 00000 | 67 |
| 852 | 21000 | 00000 | 71 |
| 876 | 21000 | 00000 | 73 |
| 948 | 21000 | 00000 | 79 |
| 372 | 21000 | 00000 | 31 |
| 348 | 21000 | 00001 | |
| 276 | 21000 | 00010 | |
| 228 | 21000 | 00100 | |
| 204 | 21000 | 01000 | |
| 156 | 21000 | 10000 | |
| 132 | 21001 | 00000 | |
| 84 | 21010 | 00000 | |
| 924 | 21011 | 00000 | |
| 588 | 21020 | 00000 | |
| 60 | 21100 | 00000 | |
| 780 | 21100 | 10000 | |
| 660 | 21101 | 00000 | |
| 420 | 21110 | 00000 | |
| 300 | 21200 | 00000 | |
| 36 | 22000 | 00000 | |
| 828 | 22000 | 00010 | |
| 684 | 22000 | 00100 | |
| 612 | 22000 | 01000 | |
| 468 | 22000 | 10000 | |
| 396 | 22001 | 00000 | |
| 252 | 22010 | 00000 | |
| 180 | 22100 | 00000 | |
| 900 | 22200 | 00000 | |
| 108 | 23000 | 00000 | |
| 756 | 23010 | 00000 | |
| 540 | 23100 | 00000 | |
| 324 | 24000 | 00000 | |

| | 1 | 11122 | |
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| 424 | 30000 | 00000 | 53 |
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| 488 | 30000 | 00000 | 61 |
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| 568 | 30000 | 00000 | 71 |
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| 776 | 30000 | 00000 | 97 |
| 808 | 30000 | 00000 | 101 |
| 824 | 30000 | 00000 | 103 |
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| 872 | 30000 | 00000 | 109 |
| 904 | 30000 | 00000 | 113 |
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| 232 | 30000 | 00001 | |
| 184 | 30000 | 00010 | |
| 152 | 30000 | 00100 | |
| 136 | 30000 | 01000 | |
| 104 | 30000 | 10000 | |
| 88 | 30001 | 00000 | |
| 56 | 30010 | 00000 | |
| 952 | 30010 | 01000 | |
| 728 | 30010 | 10000 | |
| 616 | 30011 | 00000 | |
| 392 | 30020 | 00000 | |
| 40 | 30100 | 00000 | |
| 920 | 30100 | 00010 | |
| 760 | 30100 | 00100 | |
| 680 | 30100 | 01000 | |
| 520 | 30100 | 10000 | |
| 440 | 30101 | 00000 | |
| 280 | 30110 | 00000 | |
| 200 | 30200 | 00000 | |
| 24 | 31000 | 00000 | |
| 888 | 31000 | 00000 | 37 |
| 744 | 31000 | 00000 | 31 |
| 696 | 31000 | 00001 | |
| 552 | 31000 | 00010 | |
| 456 | 31000 | 00100 | |
| 408 | 31000 | 01000 | |
| 312 | 31000 | 10000 | |
| 264 | 31001 | 00000 | |
| 168 | 31010 | 00000 | |
| 120 | 31100 | 00000 | |
| 840 | 31110 | 00000 | |
| 600 | 31200 | 00000 | |
| 72 | 32000 | 00000 | |
| 936 | 32000 | 10000 | |
| 792 | 32001 | 00000 | |
| 504 | 32010 | 00000 | |
| 360 | 32100 | 00000 | |
| 216 | 33000 | 00000 | |
| 648 | 34000 | 00000 | |

| | 1 | 11122 | |
|---|---|---|---|
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| 688 | 40000 | 00000 | 43 |
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| 848 | 40000 | 00000 | 53 |
| 944 | 40000 | 00000 | 59 |
| 496 | 40000 | 00000 | 31 |
| 464 | 40000 | 00001 | |
| 368 | 40000 | 00010 | |
| 304 | 40000 | 00100 | |
| 272 | 40000 | 01000 | |
| 208 | 40000 | 10000 | |
| 176 | 40001 | 00000 | |
| 112 | 40010 | 00000 | |
| 784 | 40020 | 00000 | |
| 80 | 40100 | 00000 | |
| 880 | 40101 | 00000 | |
| 560 | 40110 | 00000 | |
| 400 | 40200 | 00000 | |
| 48 | 41000 | 00000 | |
| 912 | 41000 | 00100 | |
| 816 | 41000 | 01000 | |
| 624 | 41000 | 10000 | |
| 528 | 41001 | 00000 | |
| 336 | 41010 | 00000 | |
| 240 | 41100 | 00000 | |
| 144 | 42000 | 00000 | |
| 720 | 42100 | 00000 | |
| 432 | 43000 | 00000 | |
| 32 | 50000 | 00000 | |
| 928 | 50000 | 00001 | |
| 736 | 50000 | 00010 | |
| 608 | 50000 | 00100 | |
| 544 | 50000 | 01000 | |
| 416 | 50000 | 10000 | |
| 352 | 50001 | 00000 | |
| 224 | 50010 | 00000 | |
| 160 | 50100 | 00000 | |
| 800 | 50200 | 00000 | |
| 96 | 51000 | 00000 | |
| 672 | 51010 | 00000 | |
| 480 | 51100 | 00000 | |
| 288 | 52000 | 00000 | |
| 864 | 53000 | 00000 | |
| 64 | 60000 | 00000 | |
| 832 | 60000 | 10000 | |
| 704 | 60001 | 00000 | |
| 448 | 60010 | 00000 | |
| 320 | 60100 | 00000 | |
| 192 | 61000 | 00000 | |
| 960 | 61100 | 00000 | |
| 576 | 62000 | 00000 | |
| 128 | 70000 | 00000 | |
| 896 | 70010 | 00000 | |
| 640 | 70100 | 00000 | |
| 384 | 71000 | 00000 | |
| 256 | 80000 | 00000 | |
| 768 | 81000 | 00000 | |
| 512 | 90000 | 00000 | |

# 17

# Logic and Sets

## 17.1 LOGIC

Logic concerns *propositions*; a proposition is any statement which may be judged to be either true or false. Thus a proposition is a function having a result limited to two values; these two values are usually referred to by the words *true* and *false* and represented by the integers 1 and 0. In other words, a proposition is a function whose range is 0 1. For example:

```
                        Proposition read as:
     X←3
     X<5                X is less than 5
1                       true

     0=3|X              X is divisible by 3
1                       true

     (X>5)∧0=3|X        X is greater than 5 and X is divisible by 3
0                       false
```

A proposition is also referred to as a *logical expression* or *logical function*. Although the term *proposition* has not been used in earlier chapters, propositions have been used freely. In fact any expression which used compression has had a left argument which is the result of a proposition, and most branches occurring in function definition incorporate propositions. Consider, for example, the functions *PR* and *BIN* from Sections 9.3 and 12.4:

```
      ∇Z←PR X
[1]   Z←(2=+/⍉0=(⍳X)∘.|⍳X)/⍳X∇

      ∇Z←BIN X
[1]   Z←,1
[2]   →3×X≥ρZ
[3]   Z←(Z,0)+(0,Z)
[4]   →2∇
```

The left argument of compression in the function *PR* contains two uses of propositions, first a comparison with zero (to determine divisibility) and then a comparison with 2 (to determine which integers have two divisors). Moreover, the branch on line 2 of the function *BIN* incorporates the proposition *X*≥ρ*Z*.

The important logical functions are the relations (< ≤ = ≥ > ≠), the functions and, or, nand, and nor (∧ ∨ ⍲ ⍱ ), complement (~), and set membership (ϵ). The relations have been used freely in the foregoing chapters, the complement was introduced in Section 8.6, and the functions and, or, nand, and nor were introduced and analyzed in Section 14.2. The set membership function will be introduced in the treatment of sets. There therefore remains little to be learned about logic except the use of the terms proposition, logical expression, and logical function. ⌹1-5

## 17.2 SETS

A set is a collection of items defined by some proposition. For example:

"The set of all positive even integers less than 15"

"The set of all positive factors of 24"

"The set of all items now lying on the desk"

"The set of all numbers occurring as elements of the vector *W* specified as *W*←2 3.5 7 8 13"

"The set of all vowels"

"The set of all letters occurring as elements of the vector *V* specified as *V*←'*AEIOU*'"

The primary question concerning sets is membership, that is, "is a given value of *X* a member of a specified set". Thus 8 is a member of the set of all positive factors of 24 but 5 and ¯8 are not. The membership of any value *X* in a set is determined by applying to *X* the proposition which defines the set. For example, the proposition defining the set of all positive factors of 24 is defined and used as follows:

```
      ∇Z←PD24 X
[1]   Z←(X>0)∧0=X|24∇
      PD24 8
1
      PD24 ¯8 5 8
0 0 1
      PD24 ⍳24
1 1 1 1 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1
      (PD24 ⍳24)/⍳24
1 2 3 4 6 8 12 24
```

⌹6-7

Certain propositions can be defined only in terms of an explicit list of the elements of the set. For example, the proposition defining the set of all vowels would have to employ the list '*AEIOU*'. Thus:

```
      ∇Z←VOWEL X
8⌹ [1]    Z←∨/X='AEIOU'∇
```

A more general proposition to determine membership in any list presented as the second argument could be defined as follows:

```
      ∇Z←X ISAMEMBEROF S
[1]    Z←∨/X=S∇
```

For example:

```
      'D' ISAMEMBEROF 'AEIOU'
0
      V←'AEIOU'
      'I' ISAMEMBEROF V
1
      W←2 3.5 7 8 13
      7 ISAMEMBEROF W
1
      'A' ISAMEMBEROF W
0
```

The foregoing proposition will not apply properly to a vector left argument and it will be more convenient to define the following (otherwise equivalent) function which does:

```
      ∇Z←X BELONGSTO S
[1]    Z←∨/X∘.=S∇
```

For example:

```
      A13←'ABCDEFGHIJKLM'
      A13 BELONGSTO V
1 0 0 0 1 0 0 0 1 0 0 0 0
      (A13 BELONGSTO V)/A13
AEI
```

The function *BELONGSTO* is an important function that will be assigned the symbol ∊ (i.e., the Greek letter epsilon).

```
      A13∊V
1 0 0 0 1 0 0 0 1 0 0 0 0
```

The function denoted by ∊ is referred to as "membership",
9-10⌹ "is a member of", or "belongs to".

The function ∊ is actually slightly more general than the function *BELONGSTO*; it applies to either matrix or vector arguments, determining for each element of the left argument whether it occurs as a member of the right argument. For example:

```
      M                      N                  S
  4    6    8   10       4  5  6        1  2  3  4  5
  6    9   12   15       7  8  9
  8   12   16   20
 10   15   20   25

      N∊M                    M∊N                S∊N
 1 0 1                   1 1 1 0        0  0  0  1  1
 0 1 1                   1 1 0 0
                         1 0 0 0
                         0 0 0 0                        ⌹11
```

Any set with a finite number of elements can therefore be conveniently represented by a vector *S* whose elements include all elements of the set; the proposition defining the corresponding set is then simply *X*∊*S*. Consequently it is often convenient to think of the vector *S* as the set itself. This is analogous to thinking of the string of characters 144 <u>as</u> the number which it represents, even though we know that this is only one of many possible representations, as discussed in Chapter 16. Therefore although we may carelessly refer to "the set *S*" it is important to realize that this notion may lead to confusion and one must always be ready to remember that *S* is not the set itself but only one of many possible representations of it.

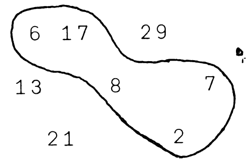
Any vector can therefore represent a set. It is, however, convenient to use only vectors which have no repeated elements, since a second occurrence of any value in the vector does not enlarge or otherwise affect membership. Thus the members of the set represented by the vector 2 3 4 6 are the same as the members of the set represented by the vector 2 3 2 4 3 6 4.

It should also be noted that any vector *T* obtained by reordering the elements of a vector *S* represents the same set as does *S*. For example:

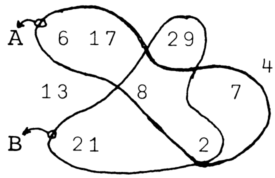
```
      A←'ABCDEFGHIJKLMNOPQRSTUVWXYZ'
      S←'STEAM'
      T←S[5 4 2 3 1]
      T
MATES
      (A∊S)/A
AEMST
      (A∊T)/A
AEMST
```

Thus the membership function $X\epsilon S$ does not depend on the order of the elements of the vector $S$ which represents the function, and for this reason it is often said that the set itself is unordered. The fact that the result of some function applied to a vector does not depend on the order of the elements in the vector is, of course, not unusual. For example, the values of the expressions $+/X$ and $\lceil/X$ do not depend on the order of the elements in $X$ (since $+$ and $\lceil$ are both associative and commutative), although the expressions
12-14⌹ $-/X$ and $\div/X$ do depend on the order.

In order to emphasize that membership in a set does not depend on the order of the elements in its representation, it is common to represent a set by distributing a number of values on a page at random and then drawing a curve around the elements belonging to the set, as follows:




If this set is labelled A and if a second curve representing the set of elements 2, 8, 29, and 21 is drawn and is labelled B, the picture appears as follows:




The intersection of the two curves (i.e., the area common to both curves) is said to represent the set which is the intersection of the sets A and B; in this case its elements are the numbers 2 and 8. A member of the intersection of the two sets must be a member of both; hence if $P$ and $Q$ are the propositions defining the two sets, the proposition defining their intersection may be stated as follows:

```
      ∇Z←R X
[1]    Z←(P X)∧(Q X)∇
```

For example:

```
      ∇Z←P X
[1]   Z←0=2|X∇

      ∇Z←Q X
[1]   Z←0=3|X∇

      P 1 2 3 4 5 6
0 1 0 1 0 1
      Q 1 2 3 4 5 6
0 0 1 0 0 1
      R 1 2 3 4 5 6
0 0 0 0 0 1
```

Moreover, if $S$ and $T$ are the vectors representing two sets, then their intersection is represented by the set $(T \in S)/T$, since the logical vector $T \in S$ selects from $T$ only those elements which also belong to $S$. A function for the intersection of two sets in terms of the vectors representing them can therefore be defined as follows:

```
      ∇Z←X I Y
[1]   Z←(X∊Y)/X∇
```

For example, the intersection of the sets $A$ and $B$ previously represented by closed curves can be determined as follows:

```
      1 16 2 17 8 24 I 8 21 2 29
2  8
```
⍠15

The set $S$ <u>less</u> the set $T$ refers to the set which contains all elements of $S$ <u>except</u> those also contained in $T$. The expression $(\sim T \in S)/T$ clearly selects from $T$ all those elements which do not belong to $S$. Hence the <u>less</u> function can be defined as follows:

```
      ∇Z←X L Y
[1]   Z←(~X∊Y)/X∇
```

For example:

```
      1 16 2 17 8 24 L 8 21 2 29
1  16  17 24
```
⍠16

The <u>union</u> of two sets refers to the set which contains all elements which are in either of the two sets. The union of the sets represented by $S$ and $T$ can therefore be represented by the vector $S,T$. However, to avoid repeated elements in the vector it is better to use the expression $S,T\ L\ S$. Thus:

```
      ∇Z←X U Y
[1]   Z←X,Y L X∇

      1 16 2 17 8 24 U 8 21 2 29
1 16  2  17  8  24  21  29
```
⍠17-18

If every element of a set $S$ belongs to a second set $T$, then $S$ is said to be a <u>subset</u> of $T$. If $E$ is any logical vector then the set $E/T$ is clearly a subset of $T$. Moreover, every possible subset of $T$ can be written as $E/T$ for a suitable choice of $E$. If $T$ is a vector of three elements, then *E* must also have three elements and every possible value of *E* is listed as some row of the following table:

```
      TAB
0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1
```

This table is precisely the table produced in Exercise 16.16 to list all possible 3-digit binary numbers in ascending order. More generally it can be seen that the logical vectors representing all possible subsets of a set of *N* elements are listed in the table of all *N*-digit binary numbers.

The transpose of the matrix *TAB* appears as follows:

```
      ⍉TAB
0 0 0 0 1 1 1 1
0 0 1 1 0 0 1 1
0 1 0 1 0 1 0 1
```

If *T* is a numeric vector of three elements, then the inner product R←T+.×⍉TAB yields a vector of the sums over all possible subsets of *T*; that is, R[1] is the sum over no elements, R[2] is the sum over the last element, R[3] is the sum over the second element, R[4] is the sum over the last two elements, and so on. For example:

```
      T←2 3 7
      T+.×⍉TAB
0  7  3  10  2  9  5  12
```

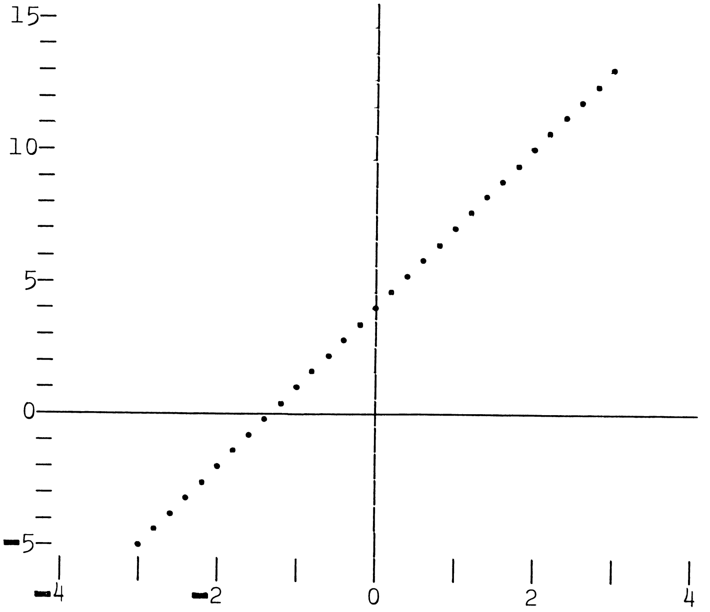
19-20⌹ (margin mark beside the last output line)

# 18

# Linear Functions

## 18.1 INTRODUCTION

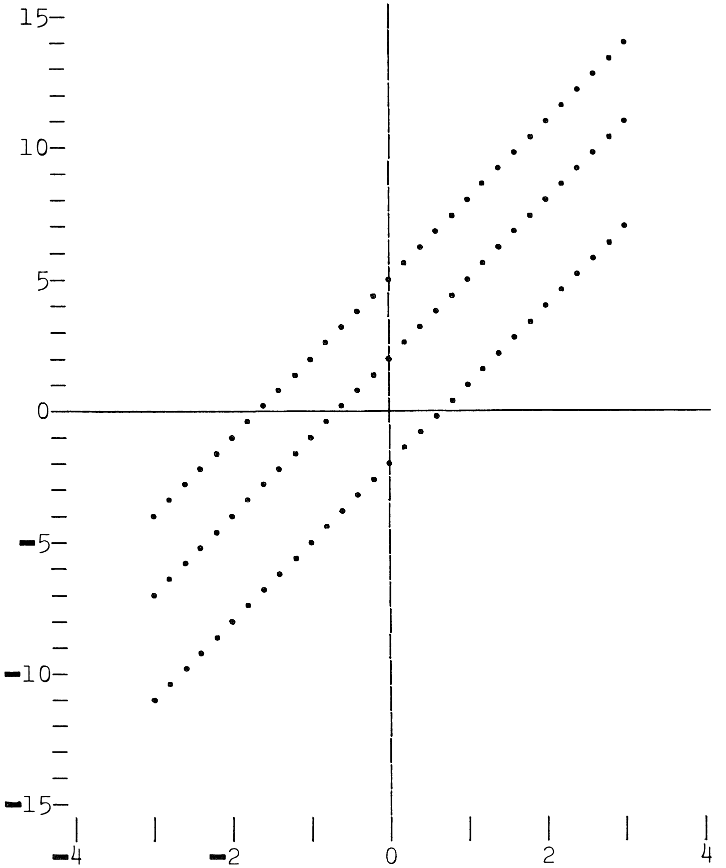
The expression 4+3×*X* is said to be a <u>linear</u> function. The reason for the term "linear" becomes evident on plotting the function; as shown in Figure 18.1, the plot forms a straight line.




The Linear Function 4+3×*X*

Figure 18.1

More generally, if *A* and *B* are any scalar constants, then the expression *A*+*B*×*X* is a linear function. A plot of several linear functions sharing the same value of *B* and having different values of *A* (Figure 18.2) shows that the graphs have the same slope (i.e., they are parallel), but that they intercept the *Y*-axis at different points determined directly by the value of *A*. That is, the *Y*-intercept of the function 5+3×*X* is 5, the *Y*-intercept of 2+3×*X* is 2, and so on.

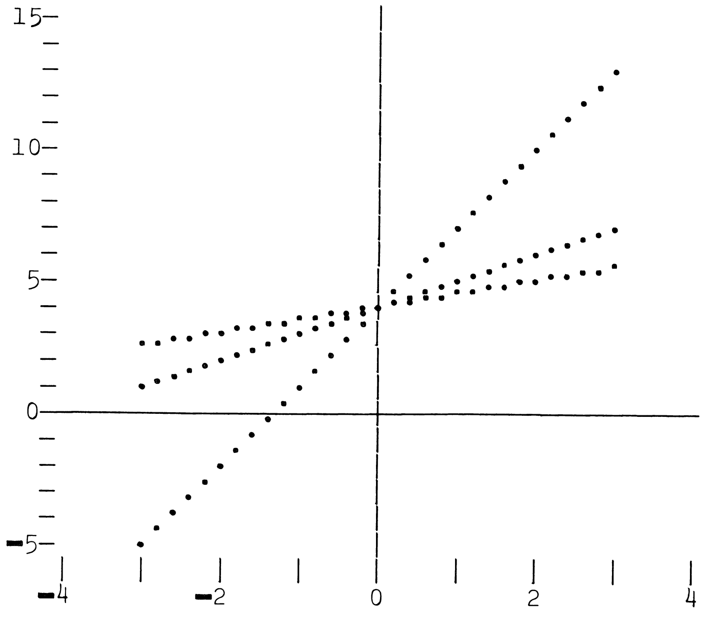



Linear Functions $A+3\times X$ (Common Slope)

Figure 18.2

A plot of the function $A+B\times X$ for a common value of $A$ and different values of $B$ (Figure 18.3) shows that the functions share the same $Y$-intercept but have different slopes which are directly determined by $B$, that is, the vertical distance between any two points on the graph is $B$ times the horizontal distance between them.

If $A$, $B$, and $C$ are scalar constants, the expression $A+(B\times X)+(C\times Y)$ is a function of two arguments $X$ and $Y$, but for any fixed value of $X$ it is a linear function of $Y$ alone. For example, the function $1+(2\times X)+(3\times Y)$ is equivalent to $1+(2\times 4)+(3\times Y)$ if $X$ is given the fixed value $4$. This in turn equals $9+3\times Y$, which is clearly a linear function of $Y$.




Linear Functions 4+*B*×*X* (Common Intercept)

Figure 18.3

Similarly, for a fixed value of *Y*, the expression *A*+(*B*×*X*)+(*C*×*Y*) is a linear function of *X*. Consequently it is said to be a <u>linear function of two arguments</u>.

If the two arguments *X* and *Y* are combined in a single two-element vector *V*, then the linear function 1+(2×*X*)+(3×*Y*) can be written more concisely as 1+2 3+.×*V*. More generally, for any scalar *A* and any two-element vector *B*, the expression *A*+*B*+.×*V* represents a linear function of the two arguments *V*[1] and *V*[2].

This vector form of writing linear equations possesses three important advantages. First, the expression *A*+*B*+.×*V* applies for a linear function of any number of arguments; it is only necessary that *B* and *V* each have the same number of elements as there are arguments. For example, the expression 1+2 3 4+.×*V* represents a linear function of the three arguments *V*[1], *V*[2], and *V*[3]. It could be written in terms of these individual arguments as follows:

```
1+(2×V[1])+(3×V[2])+(4×V[3])
```

or, if the three arguments are called *X*, *Y*, and *Z* it could be written as:

```
1+(2×X)+(3×Y)+(4×Z)
```

The second advantage of using the expression `A+B+.×V` is that it can express not only one linear function, but several. For example, if *B* is the matrix

```
      B←2 2ρ2 3 1 4
      B
  2  3
  1  4
```

and *A* is the vector `5 7`, then `A+B+.×V` yields two results:

```
      5+2 3+.×V
```

and

```
      7+1 4+.×V
```

Hence `A+B+.×V` expresses two linear functions in two arguments.

In general, if *A* is a vector of *M* elements and *B* is an *M* by *N* matrix, then `A+B+.×V` expresses *M* linear functions in
4-6⊟ *N* arguments.

## 18.2 MAPPINGS

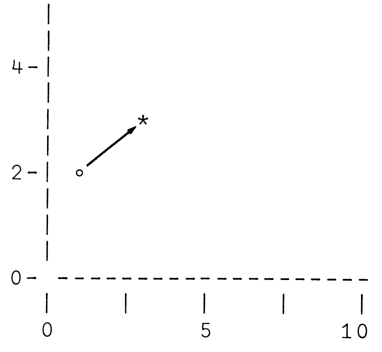
If *A* is a two-element vector and *B* is a 2 by 2 matrix, then the expression `A+B+.×V` applies to a two-element vector *V* and yields a two-element vector as a result. For example:

```
      A←¯2 ¯4
      B←2 2ρ1 2 3 2
      B
  1  2
  3  2
      B+.×1 2
  5  7
      A+B+.×1 2
  3  3
```

The vector `1 2` can be shown as a point on the graph as can the vector `3 3` which results from applying the linear function `A+B+.×V` to it. Hence the effect of the linear function can be shown as a map by drawing an arrow from the point representing the vector `1 2` to the point representing the result `3 3`. This is shown in Figure 18.4.

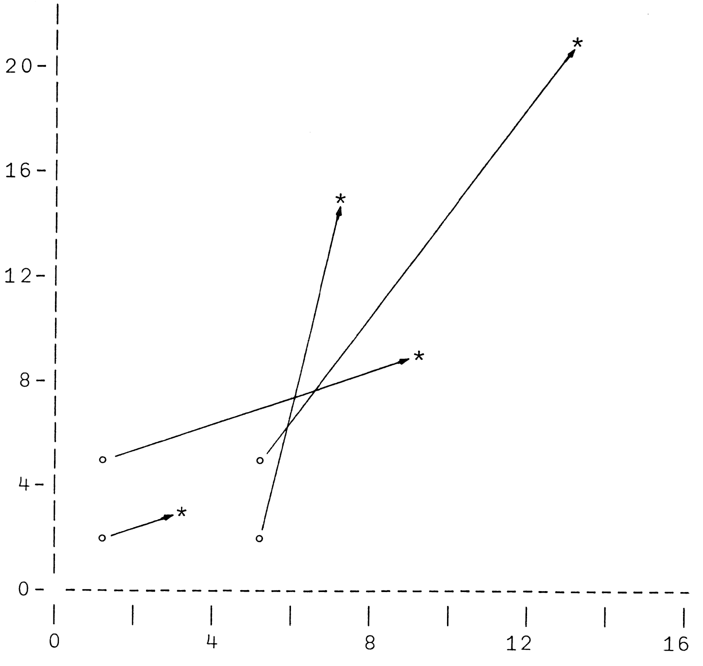
A more complete picture of the effect of the linear function `A+B+.×V` can be obtained by computing and plotting the results from applying it to a number of points. Figure 18.5 shows the mapping from the points `1 2` and `1 5` and `5 5`
7⊟ and `5 2`.

The effects of *A* and *B* can be studied separately by considering certain special cases. For example, if *A* has the value `0 0`, then `A+B+.×V` is equivalent to `B+.×V`.




A Linear Mapping

Figure 18.4



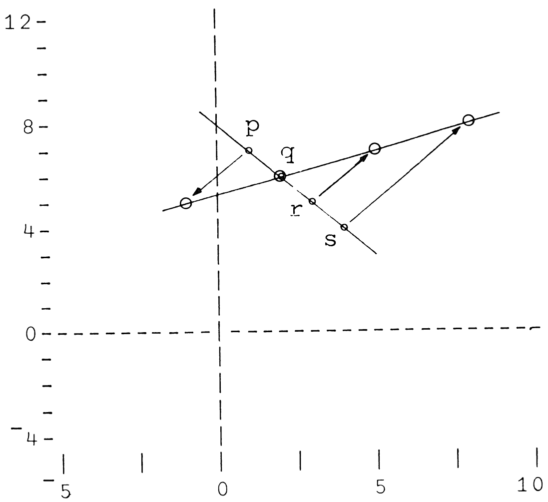

A Linear Mapping on Several Points

Figure 18.5

The linear function *B*+.×*V* always leaves the origin (the point 0 0) unchanged, that is, *B*+.×0 0 is 0 0 no matter what *B* is. Apart from this simple fact, the mapping produced by *B*+.×*V* can be quite complicated. For example, if

```
      B←2 2⍴2.5 ¯.5 1.5 .5
      B
    2.5    ¯0.5
    1.5     0.5
      B+.×1 7
¯1   5
      B+.×2 6
2   6
      B+.×3 5
5   7
      B+.×4 4
8   8
```

then the mapping produced by *B*+.×*V* is shown in Figure 18.6. From this figure it appears that the effects on different points may be quite different. For example, the last point s is "stretched" (that is, it maps into a point straight away from the origin in the same direction as s), the second point q maps into itself, and the arrows from p and r lead in opposite directions. Points (such as p, q, r, and s) which lie on a line do, as remarked before, map into points
8⌹ which also lie on a line.




A Linear Mapping

Figure 18.6

## 18.3 ROTATIONS

There is a certain class of matrices which yields a very simple and important mapping. If *B* is a 2 by 2 matrix of the form

```
 S   C
-C   S
```

and *C* is equal to either `(1-S*2)*.5` or `-(1-S*2)*.5`, then the mapping `B+.×V` is a *rotation* about the origin. That is, each point maps into a point the same distance from the origin but displaced by rotation through a certain angle. Such a matrix will be called a *rotation matrix*. For example, if `S←.5`, then `(1-S*2)*.5` is equal to `(3÷4)*.5` (which is approximately `.866`), and *B* is the matrix:

```
 0.5      0.866
¯0.866    0.5
```

Figure 18.7 shows the mapping `B+.×V` applied to the following set of points:

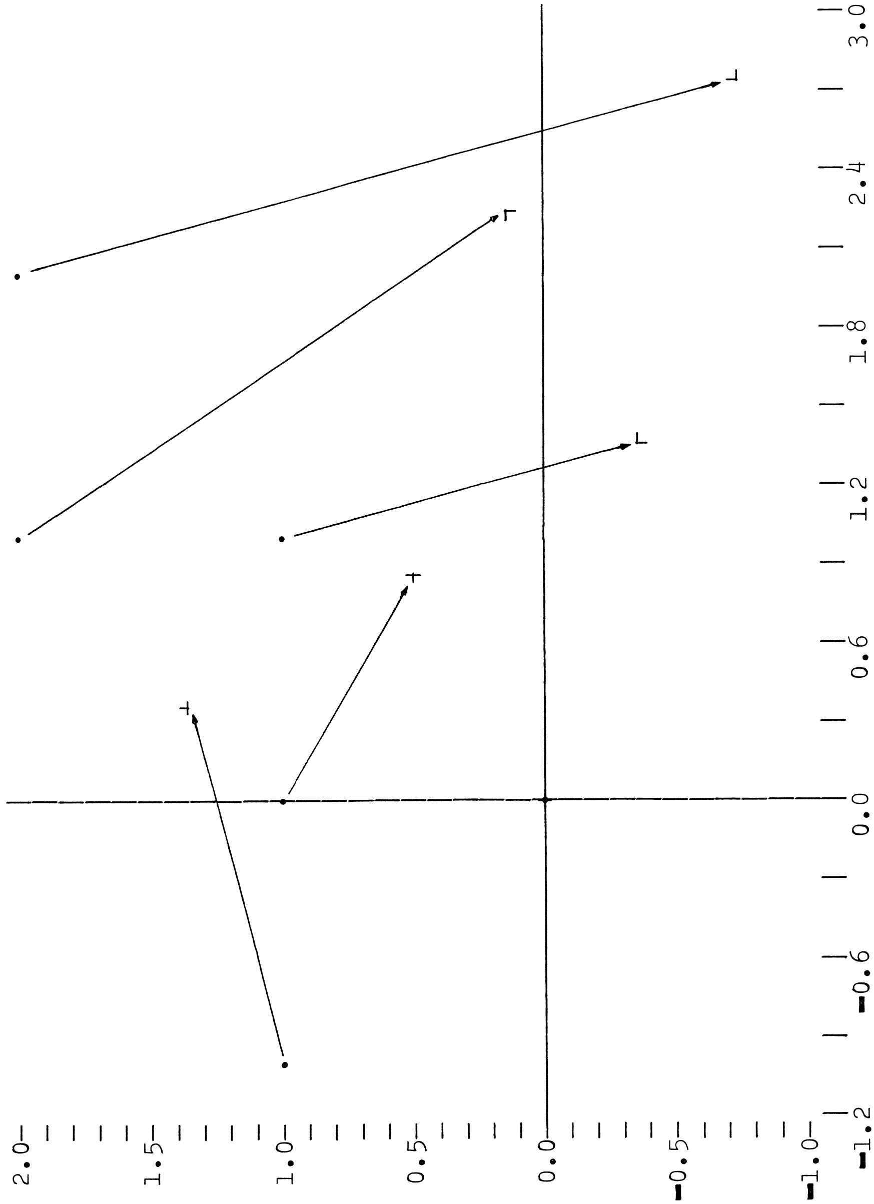
```
      B+.×0 0
0  0
      B+.×1 1
1.366  ¯0.366
      B+.×2 2
2.732  ¯0.732
      B+.×¯1 1
0.366  1.366
      B+.×0 1
0.866  0.5
      B+.×1 2
2.232  0.134
```

To see why this mapping is called a rotation, lay a sheet of translucent paper over the plot and copy onto it the original points *V* and the axes. Then place a pin through the origin and rotate the translucent overlay until one of the points *V* coincides with the point `B+.×V` into which it maps. It will then be seen that all points in *V* lie over the corresponding points `B+.×V`. Moreover, the *angle* of rotation is the angle formed between the new and old positions of the axes.

If *S* is equal to 1, then `(1-S*2)*.5` is equal to zero, and the rotation matrix *B* becomes

```
1  0
0  1
```

In this case it is clear that `B+.×V` yields *V* for any *V*. The mapping `B+.×V` is therefore called the *identity* mapping, and the matrix *B* is called the *identity* matrix. ⌹9-13



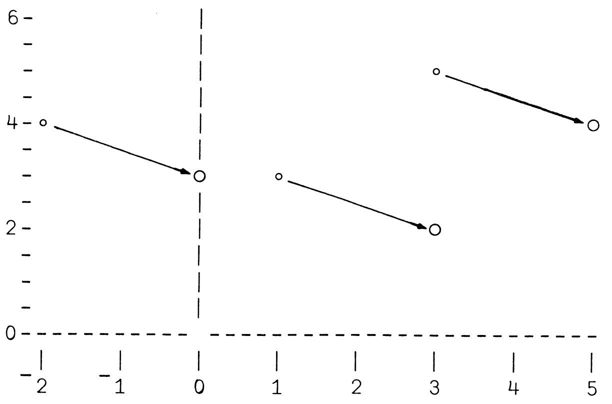
A Rotation

Figure 18.7

## 18.4 TRANSLATION

The effect of the vector *A* in the linear function *A*+*B*+.×*V* is most easily seen if *B* is chosen to be the identity matrix. In that case *B*+.×*V* yields *V* and the expression *A*+*B*+.×*V* is therefore equivalent to the expression *A*+*V*. This mapping is shown in Figure 18.8 for the case *A*←2 ¯1. All of the mapping arrows are parallel and of the same length. This sort of mapping is called a translation.

If the first element of *A* is zero, the translation is vertical, moving upward if *A*[2] is positive and downward if it is negative. Likewise, if the second element is zero the translation is horizontal, to the right if *A*[1] is positive, and to the left if it is negative. ⌹14




Translation

Figure 18.8

## 18.5 LINEAR FUNCTION ON A SET OF POINTS

It is often necessary to apply the expression *B*+.×*V* to a number of points, that is, for a number of different values of *V*. This can be done conveniently by assembling the values into a single matrix *M* such that each point appears as a column of *M*. Then the expression *B*+.×*M* yields a matrix whose columns are the results of applying the linear function to each column of *M*. For example, if the required points are 2 3 and 4 2 and 1 5, then

```
      M←⍉3 2⍴2 3, 4 2, 1 5
      M
 2  4  1
 3  2  5
```

Moreover, if

```
      B←2 2⍴1 2 3 2
      B
 1  2
 3  2
```

then

```
      B+.×M
   8   8  11
15⌹  12  16  13
```

The translation `A+V` does not extend to a matrix of points quite so neatly as does the expression `B+.×V`. For example, if `A←1 2` and `M` is the matrix of the preceding paragraph, then `A+2 3` is a translation of the vector `2 3` but `A+M` cannot be evaluated because `A` and `M` are not of the same shape. What is needed is a matrix `P` of the same shape as `M` and having each column equal to `A`, that is:

```
      P
 1  1  1
 3  3  3
```

Then `P+M` yields the desired translation of the columns of `M`;

```
      P+M
 3  5  2
 6  5  8
```

The matrix `P` can be obtained by the expression `⍉(⌽⍴M)⍴A`. Hence the translation of a set of points `M` can be expressed as:

```
      (⍉(⌽⍴M)⍴A)+M
```

and the general linear function `A+B+.×V` can be expressed for a set of points `M` as:

```
16⌹      (⍉(⌽⍴M)⍴A)+B+.×M
```

## 18.6 ROTATION AND TRANSLATION

If `B` is a rotation matrix, then the function `B+.×V` is a rotation and the function `A+B+.×V` is a rotation followed by a translation. Similarly, `B+.×A+V` is a translation followed by a rotation. A few experiments with these expressions for some chosen values of `A` and `B` applied to a number of points `V` will show that the two expressions are <u>not</u> equivalent.

However, the same experiments will be seen to suggest that `B+.×A+V` <u>is</u> equivalent to rotation by `B` (that is, `B+.×V`) followed by some translation. The amount of the translation will be found to be not *A* but rather *B*+.×*A*. In other words:

```
B+.×A+V
(B+.×A)+(B+.×V)
```

The foregoing identity expresses the fact that the inner product function +.× distributes over +. This identity holds for any matrix *B* (i.e., it is not limited to rotation matrices). A proof of this for 2 by 2 matrices is fairly simple and is outlined in an exercise. The identity also holds for matrices *B* of any dimension. The proof of this is more involved and will not be attempted here, although the reader should be able to extend the method of proof used for a 2 by 2 matrix to the case of a 3 by 3 matrix. Any reader not wishing to work through the proofs may wish to shore up his faith in the identity by performing a number of experiments. ⌹17

## 18.7 STRETCHING

If *B* is the matrix

```
3 0
0 3
```

then the expression *B*+.×*V* "stretches" the point *V* by a factor of 3, since each element of the result is 3 times the corresponding element of *V*. In a plot, such stretching is equivalent to extending the line from the origin to the point *V* to 3 times its length. If *I* is the identity matrix and *T* is any scalar value, then *T*×*I* is a stretching matrix whose degree of stretch is equal to *T*.

A more general stretching is illustrated by the matrix *B* below:

```
3 0
0 2
```

For such a matrix, the expression *B*+.×*V* stretches by a different amount for each coordinate. ⌹18

## 18.8 IDENTITIES ON THE INNER PRODUCT +.×

The inner product +.× has been seen to be central to the treatment of linear functions. Certain identities involving the inner product are also important in the study of linear functions. One of these has already been established, namely, the distributivity of +.× over +:

```
B+.×A+V
(B+.×A)+(B+.×V)
```

A second important fact is that this inner product +.× is associative, that is:

```
      M+.×(B+.×V)
      (M+.×B)+.×V
```

A proof of this will be outlined in exercises for the case
19-21⊞ of 2 by 2 matrices *M* and *B*.

## 18.9 LINEAR FUNCTIONS ON 3-ELEMENT VECTORS

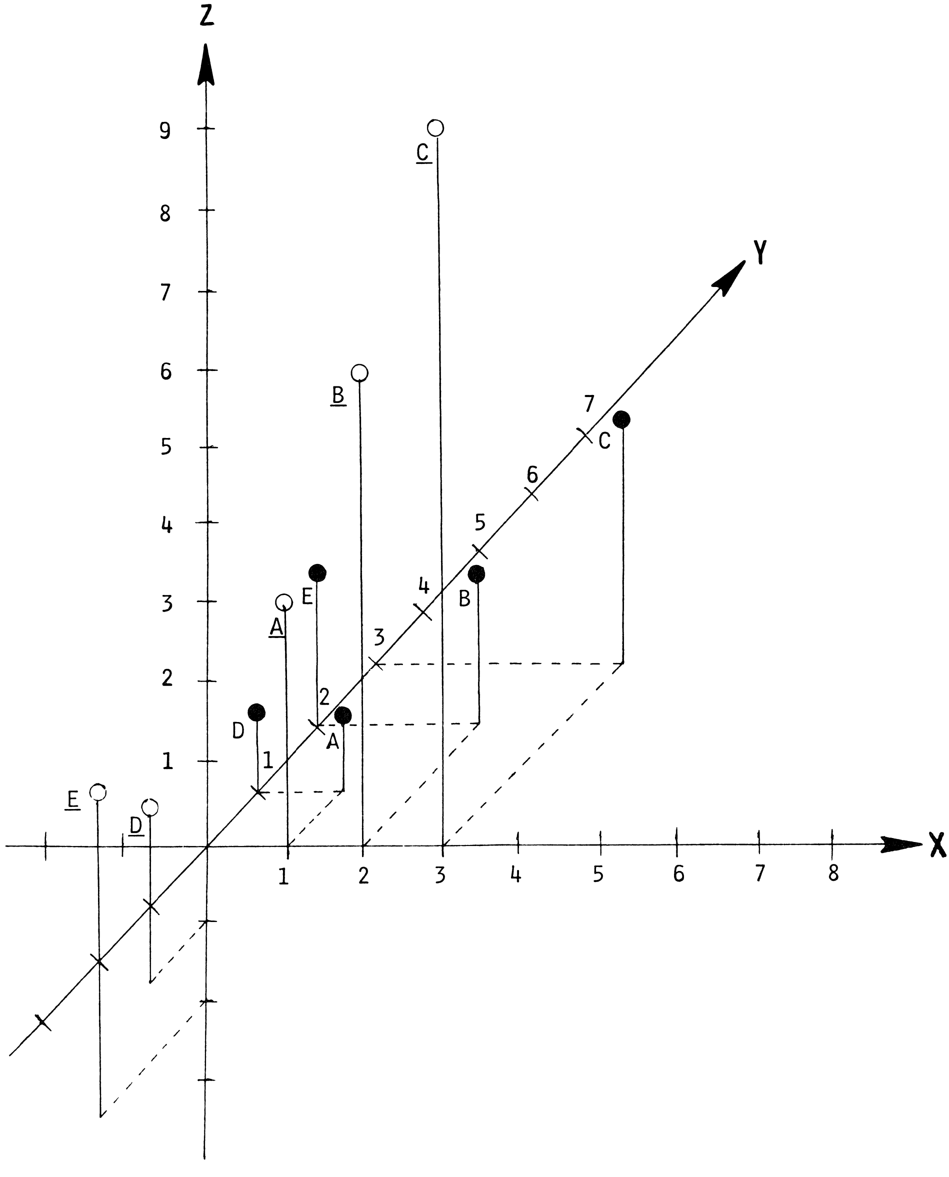
If *V* is a 3-element vector, *B* is a 3 by 3 matrix and *A* is a 3-element vector, then *A+B+.×V* is again a linear function of *V* which produces a 3-element result. In order to get a clear picture of the mapping produced by the function *A+B+.×V* for vectors *V* of dimension 3, it is necessary to devise a way of plotting a point having 3 coordinates: Draw the usual coordinates for a graph on a flat piece of thick styrofoam and obtain a set of wires of various lengths. Stick a wire into the point 3 4 on the graph so that it extends straight up to a length of 5 units. The tip of the wire then represents the point (that is, the vector) 3 4 5. Other points can be represented similarly.

The points plotted in 3-dimensions will be easier to see if the wires are tipped with colored beads. Moreover, if two different colors are used to plot the points *V* and the points *A+B+.×V*, then the effect of a linear mapping can be observed easily. Light tape can be used to connect each point to the corresponding point produced by the linear function. Alternatively, numeric labels identifying the points can be attached to them. For example:

```
      B←3 3⍴2 0 ¯1 1 ¯2 1 1 1 1
      M←⍉5 3⍴1 1 1, 2 2 2, 3 3 3, 0 1 1, 0 2 2
      B                            M
 2  0 ¯1                    1  2  3  0  0
 1 ¯2  1                    1  2  3  1  2
 1  1  1                    1  2  3  1  2
      B+.×M
 1  2  3 ¯1 ¯2
 0  0  0 ¯1 ¯2
 3  6  9  2  4
```

The plot of this mapping is shown in Figure 18.9.

Most of the properties of linear functions observed for 2-element vectors carry over to the case of 3-dimensions. For example, points lying on any line map into points lying on a line. Since this is true for a line in any direction it is also true for any plane, that is, points lying in the same plane map into points lying in a plane. Performing and plotting experiments for various values of *B* and *V* should make this clear.




A Mapping in Three Dimensions

Figure 18.9

The identity matrix for 3-dimensions is the matrix *I* shown below:

```
1 0 0
0 1 0
0 0 1
```

It is easy to show that this is the identity matrix by
22-23 showing that `I+.×V` yields *V* for any 3-element vector *V*.

## 18.10 ROTATIONS IN THREE DIMENSIONS

In an earlier section it was shown that the expression `B+.×V` produced a rotation (in two-dimensions) if *B* was a matrix of the form:

```
 S  C
-C  S
```

where *C* is equal to `(1-S*2)*.5` or to `-(1-S*2)*.5`.

It was also shown (in Exercise 18.13) that for such a matrix *B*, multiplication by its transpose yields the identity matrix, that is: `B+.×⍉B` is equal to the identity matrix. This is the essential property of a rotation matrix and applies in 3-dimensions as well. Thus any 3 by 3 matrix *B* such that `B+.×⍉B` yields the identity matrix is a rotation matrix. For example, if *S* and *C* satisfy the requirements imposed in the first paragraph, then the following matrix *R* is a rotation matrix:

```
      R                               ⍉R
1  0  0                         1  0  0
0  S  C                         0  S -C
0 -C  S                         0  C  S
```

```
      R+.×⍉R
1           0                       0
0     (S*2)+(C*2)          (S×-C)+(C×S)
0     (-C×S)+(S×C)          (C*2)+(S*2)
```

Since `(S*2)+(C*2)` equals 1, the matrix `R+.×⍉R` is the identity.

Similarly,

```
 S  C  0          S  0  C
-C  S  0   and    0  1  0
 0  0  1         -C  0  S
```

are rotation matrices. Moreover, if *R* and *T* are rotation
24-25 matrices then the product `R+.×T` is also a rotation matrix.

# 19

# Inverse Linear Functions

## 19.1 INTRODUCTION

The importance of inverse functions was noted in Chapter 11 where it was remarked that whenever one finds use for a particular function, the need for the inverse of that function usually arises. This is true of linear functions, and this chapter will be devoted to methods for obtaining the inverse of a linear function.

For a linear function of a single argument $X$, the inverse has already been determined in Chapter 11, where it was shown that the inverse of the function

```
A+B×X
```

was

```
(÷B)×(-A)+X
```

For example, if $A$ is 3 and $B$ is 4 and $X$ is 7, then `A+B×X` makes 31. Applying the inverse function to this result yields:

```
(÷4)×(-3)+31
(÷4)×28
7
```

Hence the result is the original value of $X$ as required.

An important point is that the inverse function `(÷B)×(-A)+X` is itself a linear function. To show that this is so, we write the expression in an equivalent form as follows:

```
(÷B)×(-A)+X
((÷B)×(-A))+((÷B)×X)
```

The last expression is a linear function since it is a constant (that is, `(÷B)×(-A)`) added to a constant (that is, `÷B`) times $X$. For example, if $A$ is 8 and $B$ is 4, then the original linear function `A+B×X` is `8+4×X` and the inverse is

```
((÷4)×(-8))+((÷4)×X)
¯2+.25×X
```

Chapter 11 dealt only with the inverses of functions of a single argument and, strictly speaking, the notion of inverse functions applies only to such a case. However, as shown in Chapter 18, a linear function of several arguments *X*, *Y*, and *Z* can be treated as a function of the single vector argument *V*, where *V*←*X*,*Y*,*Z*. In this sense, a linear function of several arguments does possess an inverse. As was just shown for the case of a single argument *X*, the inverse of any linear function is itself a linear function.

## 19.2 SOME INVERSE FUNCTIONS

As we did in the study of linear functions in Chapter 18, we will begin with a simple case in which *A* is zero, that is, we will consider the linear function *B*+.×*V*. Suppose that *B* and *IB* are defined as follows:

```
      B←2 2⍴3 1 5 2
      IB←2 2⍴2 ¯1 ¯5 3
      B                  IB
 3  1               2 ¯1
 5  2              ¯5  3
```

Then the linear function *IB*+.×*V* is the inverse of the function *B*+.×*V*. This can be tested on a number of examples as follows:

```
       B+.×1 2
5  9
      IB+.×5 9
1  2
      B+.×¯3 4
¯5  ¯7
      IB+.×¯5 ¯7
¯3  4
      B+.×IB+.×2 5
2  5
      IB+.×B+.×2 5
2  5
```

Similarly, in 3 dimensions the following matrices *B* and *IB* define inverse functions:

```
      B←3 3⍴1 0 2 2 1 3 4 0 4
      IB←3 3⍴¯1 0 .5 ¯1 1 ¯.25 1 0 ¯.25
      B                IB
 1  0  2         ¯1  0   .5
 2  1  3         ¯1  1  ¯.25
 4  0  4          1  0  ¯.25
      B+.×1 2 4
9  16  20
      IB+.×9 16 20
1  2  4
```

1-2

The foregoing illustrates how the linear function *B+.×V* may have an inverse *IB+.×V* which is also a linear function. It does not show how to go about finding a suitable inverse *IB* for any given matrix *B*. This is a rather difficult matter which will be addressed in subsequent sections.

In these later sections we will be considering the problem of finding an inverse for the function *B+.×V* and will ignore the more general problem of finding an inverse to the general linear function *A+B+.×V*. The reason is that the inverse to *A+B+.×V* can be easily obtained once we find an inverse to *B+.×V*. This will now be shown.

Suppose a matrix *IB* has been found which is inverse to *B*, that is,

    IB+.×B+.×V    yields V.

Then *IB+.×(-A)+V* is the function inverse to *A+B+.×V*. For:

    IB+.×(-A)+(A+B+.×V)
    IB+.×((-A)+A)+(B+.×V)      Associativity of +
    IB+.×0+(B+.×V)
    IB+.×B+.×V
    V                          Because IB is inverse of B

Consequently, attention will be restricted to the problem of finding an inverse to the function *B+.×V*.

## 19.3 THE SOLUTION OF LINEAR EQUATIONS

In Section 11.7 it was remarked that even though a general expression for a function *G* inverse to *F* could not be found, one could find the value of *G N* for any argument *N* by simply finding a value of *Y* such that

    N=F Y

This value satisfies the only requirement on *G*, namely, that *F G N* must be equal to *N*, for if *G N* is *Y*, then *F G N* is *F Y* which in turn is equal to *N* since *Y* was so chosen.

Finding a value of *Y* such that *N=F Y* is called "solving the equation *N=F Y*". It is often easier to solve such an equation than to find a general expression for the inverse function *G*. Moreover, solving such an equation for several different values of *N* may give some clues to an expression for *G*.

In any case, we shall approach the problem of finding an inverse to the function *B+.×V* by developing methods for solving the equation *N=B+.×V*. Since *N* is a vector, we require a value of *V* such that <u>each</u> element of *N* agrees with each element of `B+.×V`. This can be expressed by saying that the following expression is required to be true:

```
      ∧/N=B+.×V
```

For example, if

```
      B←2 2⍴1 2 2 3
      B
 1  2
 2  3
      N←3 4
      V←1 1
      B+.×V
3  5
      N=B+.×V
1  0
      ∧/N=B+.×V
0
```

then the first element of `B+.×V` agrees with the first element of `N`, but `V` is not a solution of the equation `N=B+.×V` since the elements do not all agree, as shown by the zero value resulting from the expression `∧/N=B+.×V`. However, the vector `¯1 2` is a solution as shown below:

```
      V←¯1 2
      B+.×V
3  4
      N=B+.×V
1  1
      ∧/N=B+.×V
1
```

3-4

## 19.4 BASIC SOLUTIONS

A solution of the equation

```
      ∧/1 0 = B+.×V
```

or of the equation

```
      ∧/0 1 = B+.×V
```

will be called a basic solution. Basic solutions have two important properties:

They are rather easy to obtain.

They can be used to determine solutions to the equation `∧/N=B+.×V` for any value of `N`.

The second matter will be explored first, that is, we will first assume that we know two basic solutions *V1* and *V2* such that

```
      ∧/1 0=B+.×V1
      ∧/0 1=B+.×V2
```

and will show how *V1* and *V2* can be used to determine a solution to the general equation `∧/N=B+.×V`. The matter of how to determine *V1* and *V2* themselves will be deferred to the succeeding section.

If *V1* and *V2* are basic solutions for a matrix *B*, then the vector

```
      V←(N[1]×V1)+(N[2]×V2)
```

is a solution of the equation `∧/N=B+.×V`. For example, if *B* is the matrix

```
 4  2
 1  3
```

then

```
      V1←.3 ¯.1
      V2←¯.2  .4
```

are basic solutions, for:

```
      B+.×V1
 1 0
      B+.×V2
 0 1
```

Moreover, if `N←3 5`, then:

```
      V←(N[1] ×V1)+(N[2] ×V2)
      V
¯0.1  1.7
      B+.×V
3  5
```

and *V* is indeed a solution of the equation `∧/N=B+.×V`. ⌹5-6

The method is based on two simple facts:

1) `B+.×S×V` is equal to `S×B+.×V` for any scalar *S*

2) `B+.×P+Q` is equal to `(B+.×P)+(B+.×Q)`
   (Distributivity of `+.×` over `+`)

The first of these facts is easily established and the second was established in Exercises in Chapter 18.

The following arguments can now be used to show that `V←(N[1]×V1)+(N[2]×V2)` is in fact a solution of the equation `∧/N=B+.×V`:

```
B+.×V
B+.×((N[1]×V1)+(N[2]×V2))      Definition of V
(B+.×N[1]×V1)+(B+.×N[2]×V2)    Fact 2
(N[1]×B+.×V1)+(N[2]×B+.×V2)    Fact 1
(N[1]×1 0)+(N[2]×0 1)          Definition of V1 and V2
(N[1],0)+(0,N[2])
N
```

## 19.5 DETERMINING BASIC SOLUTIONS

We now address the problem of finding basic solutions, that is, finding solutions `V1` and `V2` for the following set of equations:

```
∧/1 0 = B+.×V1
∧/0 1 = B+.×V2
```

If one has a vector `VA` such that `B+.×VA` is equal to `S,0` then `V1←(÷S)×VA` is a basic solution. For example:

```
      B
 1  3
 4  2
      VA
 2 ¯4
      B+.×VA
¯10 0
      V1←(÷¯10)×VA
      V1
¯.2  .4
      B+.×V1
 1  0
```

The foregoing is a simple application of Fact 1 of the preceding section. Moreover, the expression `(÷S)×VA` can be written equivalently as `VA÷S`.

To find a basic solution we can therefore begin with the simpler problem of finding a vector `VA` such that `B+.×VA` is equal to `S,0` for any value of `S`. It is easy to choose a value of `VA` such that the second element of `B+.×VA` is zero; simply take the second row of `B`, reverse the sign of its first element, and then reverse the order of its elements. In other words:

```
      VA←⌽ ¯1 1×B[2;]
```

For example, if *B* is the matrix

```
  1  3
  4  2
```

then

```
  4  2          Second row of B (that is, B[2;])
 ¯4  2          Reversal of sign (¯1 1×B[2;])
  2 ¯4          Reversal of order (⌽¯1 1×B[2;])
```

```
      B+.×2 ¯4
¯10 0
```

Hence if `VA←2 ¯4`, then `B+.×VA` is `¯10 0`. Moreover, `V1←VA÷¯10` is a basic solution:

```
      V1←VA÷¯10
      V1
¯.2  .4
      B+.×V1
 1  0
```
⌹7-9

The following set of equivalent statements show why the second element of `B+.×VA` is zero when `VA` is determined by the foregoing procedure:

```
 (B+.×VA)[2]                    Second element of B+.×VA
 B[2;]+.×VA                     Definition of inner product
 +/B[2;]×VA                     Definition of inner product
 +/B[2;]×⌽¯1 1×B[2;]            Choice of VA
 +/B[2;]×B[2;2],-B[2;1]         Reversals of sign and order
 +/(B[2;1],B[2;2])×(B[2;2],-B[2;1])
 (B[2;1]×B[2;2])+(B[2;2]×-B[2;1])
 0
```

The entire procedure for determining the basic solution *V*1 can therefore be summarized as follows:

```
      VA←⌽¯1 1×B[2;]
      R1←B+.×VA
      V1←VA÷R1[1]
```

It should be clear that a similar procedure applies to the second basic solution *V*2 such that `∧/0 1 = B+.×V2`. It is only necessary to interchange the roles of the first and second elements as may be seen by comparing the pair of procedures below:

```
      VA←⌽¯1 1×B[2;]             VB←⌽1 ¯1×B[1;]
      R1←B+.×VA                  R2←B+.×VB
      V1←VA÷R1[1]                V2←VB÷R2[2]
```

```
      B
 3  5
 2  4
      VA←⌽¯1 1×B[2;]
      VA
4  ¯2
      R1←B+.×VA
      R1
2  0
      V1←VA÷R1[1]
      V1
2  ¯1
      B+.×V1
1  0
```

10⊞

```
      VB←⌽1 ¯1×B[1;]
      VB
¯5  3
      R2←B+.×VB
      R2
0  2
      V2←VB÷R2[2]
      V2
¯2.5  1.5
      B+.×V2
0  1
```

## 19.6 SIMPLIFIED CALCULATIONS FOR BASIC SOLUTIONS

Examination of the procedures for determining basic solutions shows that certain simplifications can be made. For example, in calculating `R1←B+.×VA`, only the first element of `R1` need be calculated since it is the only one used in the expression `V1←VA÷R1[1]`. Thus `R1[1]` can be computed as `B[1;]+.×VA`, which requires only half as much computing as does `B+.×VA`. On the other hand, it may be wise to do the whole calculation `B+.×VA` since the value of the second element (which must be zero if `VA` has been computed correctly) is a check on the work thus far.

Similar remarks apply to the calculation of `R2[2]` for the second basic solution; that is, `R2[2]` is `B[2;]+.×VB`. Moreover, `R2[2]` need not be computed at all since it is equal to `R1[1]`, as you may have noticed in previous examples and exercises. The reason for this appears in the following identity, in which the first line is the expression for `R1[1]` and the second line is the expression for `R2[2]`:

```
      +/(B[1;1],B[1;2])×(B[2;2],-B[2;1])
      +/(B[2;1],B[2;2])×((-B[1;2]),B[1;1])
```

Taking either of these expressions for `R1[1]`, it is clear that if `B` is a matrix having the elements `P`, `Q`, `R`, and `S` as follows:

```
      P  Q
      R  S
```

then `R1[1]` is equal to `(P×S)-(Q×R)`. In other words, one takes the product of the first element with the one diagonally opposite and subtracts from it the product of the remaining two elements. For example, if `B` is the matrix

```
 5  2
 7  4
```

then the value of `R1[1]` is `(5×4)-(2×7)`, that is, `6`.

Continuing with this example, the whole computation of *V*1 can be expressed as follows:

```
V1←4 ¯7÷(5×4)-(2×7)
```

Similarly, *V*2 is obtained as follows:

```
V2←¯2 5÷(5×4)-(2×7)
```

⌹11

## 19.7 THE DETERMINANT FUNCTION

The expression for *R*1[1] (or for *R*2[2]) developed in the preceding section is a very important function called the <u>determinant</u>. It was also shown that if *B* is the matrix

```
P  Q
R  S
```

then the determinant of *B* is the expression (*P*×*S*)-(*Q*×*R*).

The determinant function may be defined formally as follows:

```
    ∇ Z←DET B
[1]   Z←(B[1;1]×B[2;2])-(B[1;2]×B[2;1])∇
```

For example:

```
      B←2 2ρ5 2 7 4
      B
 5  2
 7  4
      DET B
6
```

The function *DET* will be used throughout the remainder of this chapter. The notion of determinant is used for square matrices of dimensions higher than 2 by 2, but it must be emphasized that the function *DET* applies only to 2 by 2 matrices.

⌹12-15

## 19.8 MATRIX FORM OF THE BASIC SOLUTIONS

It is convenient to represent the basic solutions *V*1 and *V*2 as a single matrix *BS* whose first column is *V*1 and whose second column is *V*2. For example, if *B* is the matrix

```
3  5
2  4
```

then *V*1←2 ¯1 and *V*2←¯2.5 1.5 and the matrix *BS* is

```
 2  ¯2.5
¯1   1.5
```

Since `B+.×V1` is `1 0`, the first column of `B+.×BS` is `1 0` and similarly the second column is `0 1`. Thus

```
      B+.×BS
  1  0
  0  1
```

Recalling the names *VA* and *VB* used in first deriving basic solutions:

```
      VA←⌽¯1  1×B[2;]
      VB←⌽ 1 ¯1×B[1;]
```

and the fact that *V1* and *V2* are obtained by dividing these vectors by the determinant of *B*:

```
      V1←VA÷DET B
      V2←VB÷DET B
```

Then if *M* is the matrix whose columns are the vectors *VA* and *VB*, it follows that the matrix *BS* of basic solutions can be obtained from *M* as follows:

```
      BS←M÷DET B
```

The matrix *M* can be determined as follows. Suppose that the elements of *B* are called *P*, *Q*, *R*, and *S* as follows:

```
      P   Q
      R   S
```

then the first column of *M* is `(S,-R)` and the second column is `((-Q),P)`. Hence *M* is

```
      S   -Q
     -R    P
```

In other words *M* is obtained from *B* by simply interchanging the first element of *B* with the one diagonally opposite, and reversing the signs of the remaining two elements. Finally, the matrix of basic solutions *BS* is obtained by dividing *M* by the determinant of *B*.

To summarize, if *B* is the matrix

```
     P   Q
     R   S
```

form the matrix

```
     S  -Q
    -R   P
```

and divide it by the determinant `(P×S)-(Q×R)` to obtain the matrix of basic solutions.

For example:

```
        B            M           DET B          BS
     9   8       6  ¯8          ¯10          ¯.6  .8
     8   6      ¯8   9                        .8 ¯.9

      B+.×BS
 1  0
 0  1
```

⌹16-17

## 19.9 THE GENERAL SOLUTION FROM THE BASIC SOLUTIONS

In section 19.4 we saw that the solution of the general linear equation

```
      ∧/N=B+.×V
```

could be obtained from the basic solutions *V*1 and *V*2 as follows:

```
      V←(N[1]×V1)+(N[2]×V2)
```

This can be written more neatly in terms of the matrix of basic solutions *BS* as follows:

```
      V←BS+.×N
```

For example, if

```
      N←5  6
      V1←2  3
      V2←4  5
```

then *BS* is

```
      2  4
      3  5
```

and

```
      N[1] ×V1
10  15
      N[2] ×V2
24  30
      (N[1] ×V1)+(N[2] ×V2)
34  45
      BS+.×N
34  45
```

⌹18-19

We will now show that *BS*+.×*N* is equivalent to (*N*[1]×*V*1)+(*N*[2]×*V*2) by showing that each of their two elements agree. Beginning with the first element:

```
(BS+.×N)[1]
BS[1;]+.×N                          Definition of inner product
(BS[1;1]×N[1])+(BS[1;2]×N[2])       Definition of inner product
(V1[1]×N[1])+(V2[1]×N[2])           Definition of BS
(N[1]×V1[1])+(N[2]×V2[1])           Commutativity of ×
((N[1]×V1)+(N[2]×V2))[1]            Definition of indexing
```

A similar proof applies for the second element.

## 19.10 THE INVERSE LINEAR FUNCTION

In the preceeding section we saw that if *BS* is the matrix of basic solutions for the matrix *B*, then *BS*+.×*N* is a solution of the general equation

```
∧/N=B+.×V
```

Consequently if *V* is any vector and *N*←*B*+.×*V* then *BS*+.×*N* yields *V*. In other words

```
BS+.×(B+.×V)
```

yields *V*. Therefore the function *BS*+.×*V* is the linear function inverse to the function *B*+.×*V*.

Since the inverse relationship is mutual, the expression

```
B+.×(BS+.×V)
```

20-21⍠ also yields *V*.

## 19.11 PROPERTIES OF THE INVERSE LINEAR FUNCTION

As noted in the preceding section

```
BS+.×(B+.×V)        B+.×(BS+.×V)
V                   V
```

Since the inner product +.× is associative, it also follows that

```
(BS+.×B)+.×V
(B+.×BS)+.×V
V
```

But the only matrix which multiplied by any vector `V` yields `V` is the identity matrix `I` which has the value

```
1  0
0  1
```

Hence

```
BS+.×B
B+.×BS
I
```

It is already clear that `B+.×BS` yields the identity matrix, since the columns of `BS` are the basic solutions for `B` and the columns of `B+.×BS` are therefore `1 0` and `0 1`. The reader may wish to verify that `BS+.×B` is also equal to the identity matrix for each of the corresponding values of `BS` and `B` determined in earlier examples and exercises. ⌹22-23

## 19.12 ALTERNATIVE DERIVATION OF THE INVERSE

The linear function `BS+.×V` inverse to `B+.×V` was first determined by computing `BS` as the matrix of basic solutions for `B`. The method used applies only for vectors `V` of dimension `2` and cannot be applied for higher dimensions. We will now develop an alternative method which is somewhat more difficult but which has the important advantage that it applies to higher dimensions.

Since `BS+.×V` is inverse to `B+.×V` only if `BS+.×B` is the identity matrix, we can pose the problem as follows: find a matrix `BS` such that `BS+.×B` is the identity matrix. We will determine `BS` in several steps. Thus if `H1` is a matrix such that `H1+.×B` is "closer" to the identity than `B` itself, we may find a second matrix `H2` such that `H2+.×(H1+.×B)` is even closer to the identity. Suppose that in four such steps the result

```
H4+.×(H3+.×(H2+.×(H1+.×B)))
```

is equal to the identity matrix. Then (because `+.×` is associative):

```
(H4+.×H3+.×H2+.×H1)+.×B
```

is also equal to the identity matrix. Hence

```
BS←H4+.×H3+.×H2+.×H1
```

is the required inverse matrix.

For example:

```
      B←2 2 ρ 5 3 4 2
      B
 5  3
 4  2
      H1←2 2 ρ .2 0 0 1
      H1
.2  0
 0  1
      H1+.×B
 1 .6
 4  2
      H2←2 2 ρ 1 0 ¯4 1
      H2
 1  0
¯4  1
      H2+.×(H1+.×B)
 1  .6
 0 ¯.4
      H3←2 2 ρ 1 0 0 ¯2.5
      H3
 1   0
 0 ¯2.5
      H3+.×(H2+.×(H1+.×B))
 1 .6
 0  1
      H4←2 2 ρ 1 ¯.6 0 1
      H4
 1 ¯.6
 0   1
      H4+.×(H3+.×(H2+.×(H1+.×B)))
 1  0
 0  1
      BS←H4+.×H3+.×H2+.×H1
      BS
¯1  1.5
 2 ¯2.5
      BS+.×B
 1  0
 0  1
```

There are a number of points to observe in the foregoing sequence. Each of the *H* matrices itself differs from the identity in only one element. *H*1+.×*B* is closer to the identity than *B* in the sense that the first element is 1; thus the first element of *H*1 was chosen as the reciprocal of 5 so as to divide the first row of *B* by 5.

The matrix *H*2 was chosen so that the second row of the result would be obtained by adding ¯4 times the first row to the second row, thus making the first element in the second row of the result zero. Thus the element `H2[2;1]` was chosen as `-(H1+.×B)[2;1]`. The result `H2+.×(H1+.×B)` therefore agrees with the identity in the entire first column.

The matrices `H3` and `H4` are chosen similarly to make the second column agree; `H3` multiplies the second row by the reciprocal of the last element of the matrix `H2+.×H1+.×B`, and `H4` adds the appropriate multiple of the second row to the first so as to make the upper right element of the result zero.

It will be instructive to repeat the foregoing sequence using a name `BT` for the intermediate results produced so that we write `BT←B` and `BT←H1+.×BT` and `BT←H2+.×BT`, etc. Moreover, if we first set `BS` to be the identity matrix, and then write `BS←H1+.×BS` and `BS←H2+.×BS`, etc., the final value of `BS` will be the required product of the `H` matrices. Thus:

```
      BT←B
      BT
 5  3
 4  2

      BT←H1+.×BT
      BT
 1  .6
 4  2
      BT←H2+.×BT
      BT
1  .6
0 ¯.4
      BT←H3+.×BT
      BT
1  .6
0  1
      BT←H4+.×BT
      BT
1  0
0  1
      BS+.×B
1  0
0  1
```

```
      BS←2 2 ρ 1 0 0 1
      BS
 1  0
 0  1

      BS←H1+.×BS
      BS
 .2  0
 0   1
      BS←H2+.×BS
      BS
 .2   0
¯.8   1
      BS←H3+.×BS
      BS
 .2   0
 2  ¯2.5
      BS←H4+.×BS
      BS
¯1   1.5
 2  ¯2.5
```

Finally, since `BS` and `BT` are subjected to the same sequence of multiplications, we can combine the matrices `BT` and *BS* into a single matrix *M* whose first two columns represent *BT* and whose last two columns represent *BS*. The foregoing computation then appears as follows:

```
      I←2 2 ρ 1 0 0 1
      I
1   0
0   1
      M←B,I
      M
5   3    1    0
4   2    0    1
      M←H1+.×M
      M
1  .6   .2    0
4   2    0    1
      M←H2+.×M
      M
1  .6   .2    0
0 ¯.4  ¯.8    1
      M←H3+.×M
      M
1  .6   .2    0
0   1    2 ¯2.5
      M←H4+.×M
      M
1   0   ¯1  1.5
0   1    2 ¯2.5
```

The last two columns of *M* are the required inverse.

In other words, if we append the identity matrix to the right of *B* and multiply the resulting matrix by any sequence of matrices such that the first two columns become the identity matrix, then the last two columns will be the inverse of the matrix *B*.

It may be noted that each of the matrices *H* were chosen such that each multiplication *H+.×M* affected only one row and affected that row in one of two simple ways:

> It multiplied the row by a scalar (chosen so as to make the diagonal element of the row equal to 1.
>
> It added to the row some multiple of another row (chosen so as to make one of the elements zero).

We can perform such a sequence of calculations without actually writing out the matrices *H* which produce them. To illustrate this we repeat the preceding example in this form together with notes showing what calculations were performed:

```
      B,I
5     3     1     0
4     2     0     1

1    .6    .2     0         Row 1 is multiplied by ÷5
4     2     0     1

1    .6    .2     0
0   ¯.4   ¯.8     1         ¯4 times row 1 is added to row 2

1    .6    .2     0
0     1     2  ¯2.5         Row 2 is multiplied by ÷¯.4

1     0    ¯1   1.5         ¯.6 times row 2 is added to row 1
0     1     2  ¯2.5
```

The foregoing should be compared carefully with the earlier example which used the matrices *H*1, *H*2, etc. This method for determining the inverse of a matrix is called the Gauss-Jordan method. ⌹24

## 19.13 EFFICIENT SOLUTION OF A LINEAR EQUATION

A solution to the equation ∧/*N*=*B*+.×*V* can be obtained by determining the matrix *BS* which is inverse to *B* and then computing *V*←*BS*+.×*N* to obtain the solution. A modification of the Gauss-Jordan method can provide the solution more efficiently as follows: apply the Guass-Jordan method to the matrix *B*,*N* instead of to *B*,*I* and the last column of the result will be the desired solution. For example, if *N* is the vector 4 6 and *B* is the matrix of the preceding example, then:

```
     B,N
5    3    4
4    2    6

1   .6   .8
4    2    6

1   .6   .8
0  ¯.4  2.8

1   .6   .8
0    1   ¯7

1    0    5
0    1   ¯7
```

The solution is therefore `5 ¯7`. This may be checked as follows:

```
      B+.×5 ¯7
4  6
      N
```
25 
```
4  6
```

## 19.14 INVERSE LINEAR FUNCTIONS IN THREE DIMENSIONS

If *V* is a vector of 3 elements and *B* is a 3 by 3 matrix, then `B+.×V` is a linear function of *V*. The inverse function `BS+.×V` can be determined by the Gauss-Jordan method. The reason it works is the same as in the case of two elements, namely, if *B* is multiplied by a sequence of matrices until the result becomes the identity matrix, then the product of that sequence of matrices is a matrix *BS* such that `BS+.×B` is the identity. In other words, *BS* is the inverse of *B*. The Gauss-Jordan method is simply an efficient way of keeping track of the product of the sequence of matrices applied to *B*.

The general scheme is to first reduce the first column to `1 0 0`, then reduce the second column to `0 1 0`, then the third column to `0 0 1`. The first operation for the first column is to divide the first row by its first element. The next is to add a multiple of the first row to the second, and the next is to add a multiple of the first row to the third. On the second column we first divide the second row by its second element and then add multiples of it to rows 1 and 3. On the third column we first divide the third row by its third element and then add multiples to rows 1 and 2. For example:

```
      B←3 3⍴2 1 3 1 0 2 4 0 4
      B
2    1    3
1    0    2
4    0    4
      B,3 3⍴1 0 0 0 1 0 0 0 1
2    1    3    1    0    0
1    0    2    0    1    0
4    0    4    0    0    1

1   .5  1.5   .5    0    0        Multiply row 1 by ÷2
1    0    2    0    1    0
4    0    4    0    0    1

1   .5  1.5   .5    0    0
0  ¯.5   .5  ¯.5    1    0        Add ¯1 times row 1 to row 2
0   ¯2   ¯2   ¯2    0    1        Add ¯4 times row 1 to row 3
```

```
1    .5   1.5   .5    0    0
0    1   ¯1     1    ¯2    0        Multiply row 2 by ÷¯.5
0   ¯2   ¯2    ¯2     0    1

1    0    2     0     1    0        Add ¯.5 times row 2 to row 1
0    1   ¯1     1    ¯2    0
0    0   ¯4     0    ¯4    1        Add 2 times row 2 to row 3

1    0    2     0     1    0
0    1   ¯1     1    ¯2    0
0    0    1     0     1  ¯.25       Multiply row 3 by ÷¯4

1    0    0     0    ¯1   .5        Add ¯2 times row 3 to row 1
0    1    0     1    ¯1  ¯.25       Add 1 times row 3 to row 2
0    0    1     0     1  ¯.25
```

The desired inverse is in the last 3 columns, that is:

```
BS←3 3⍴0 ¯1 .5 1 ¯1 ¯.25 0 1 ¯.25
      BS
0   ¯1   .5
1   ¯1  ¯.25
0    1  ¯.25
      BS+.×B
1    0    0
0    1    0
0    0    1
      B+.×BS
1    0    0
0    1    0
0    0    1
```

⌹26-28

## 19.15 THE INVERSE FUNCTION

We have seen that if *BS*+.×*B* is the identity matrix, then the function *BS*+.×*V* is inverse to the function *B*+.×*V*. For this reason the matrix *BS* is said to be the inverse of the matrix *B*. The inverse of a matrix is an important function which will be assigned the symbol ⌹. Thus if *P*←⌹*Q*, then *P*+.×*Q* and *Q*+.×*P* are both equal to the identity matrix.

Moreover, (⌹*Q*)+.×*N* is the solution of the equation ∧/*N*=*Q*+.×*V*. This is easily seen by substituting the solution (⌹*Q*)+.×*N* for *V* obtaining:

```
∧/N=Q+.×(⌹Q)+.×N
∧/N=(Q+.×⌹Q)+.×N        Associativity of +.×
∧/N=I+.×N               Q+.×⌹Q is the identity I
∧/N=N
1
```

The solution of the equation ∧/*N*=*Q*+.×*V* is also an important function of *N* and *Q* and will be assigned the symbol ⌹ as a dyadic function; that is, *N*⌹*Q* yields the solution of the equation ∧/*N*=*Q*+.×*V*. In other words:

```
N⌹Q
(⌹Q)+.×N
```

29-32⌹

## 19.16 CURVE FITTING

In Chapter 10, the problem of fitting a function *F* was posed as follows: given a table of a vector of arguments *X* and the corresponding vector *Y*←*F X*, determine a function *E* defined by some expression such that *E X* is equal to *Y*. In Chapter 10 this problem was solved by constructing a difference table and using its first row to determine multipliers of factorial polynomials whose sum became the required expression. This solution applied only to a set of arguments *X* of the form 0,⍳*N*.

In Chapter 11 the method was extended to apply to any set of equally spaced arguments, that is, to any set of arguments *X* of the form *A*+*B*×⍳*N*. Moreover, in Chapter 14 a simpler equivalent expression was found which involved a polynomial rather than the factorial polynomials. However, the method still applied only to equally spaced arguments.

The inverse linear function can now be applied in a simple manner to obtain a solution for any set of arguments *X*. We seek a vector of coefficients *C* such that the polynomial *C POL X* is equal to the required set of function values *Y*, that is:

```
∧/Y=C POL X
```

Recalling the definition of the polynomial function from Section 13.6, this requirement may be written as follows:

```
∧/Y=(X∘.*¯1+⍳⍴,C)+.×C
```

Furthermore, because *C* must have the same number of elements as *X*, the expression ⍳⍴*C* may be replaced by ⍳⍴*X* so that the outer product in the foregoing expression becomes a function of *X* only. Thus:

```
∧/Y=(X∘.*¯1+⍳⍴,X)+.×C
```

This is clearly a linear equation with a given value of *Y*, a given matrix *X*∘.*¯1+⍳⍴,*X*, and an argument *C* whose values are to be determined. Hence the required value of *C* is given by the expression:

```
Y⌹(X∘.*¯1+⍳⍴,X)
```

For example, if `X←0 3 4 6 8` (not equally spaced) and if *F* is the function `+/(⍳X)*3`, then *Y* has the value `0 36 100 441 1296`, and the square matrix `X∘.*¯1+⍳⍴X` has the value:

```
1     0     0     0      0
1     3     9    27     81
1     4    16    64    256
1     6    36   216   1296
1     8    64   512   4096
```

The solution may then be obtained by appending the vector *Y* as a final column on this matrix and applying the efficient method of Section 16.13 to the resulting matrix shown below:

```
1     0     0     0      0      0
1     3     9    27     81     36
1     4    16    64    256    100
1     6    36   216   1296    441
1     8    64   512   4096   1296
```

The solution is:

```
C←0 0 0.25 0.5 0.25
```

This result may be checked by evaluating the polynomial `C P 0 3 4 6 8`. ⌹33-34

# Exercises

1.1 Evaluate the following expressions, entering the result in the position indicated by the underscore:

$(3+4)\times 6$ ___

$3+(4\times 6)$ ___

$3+(4+6)$ ___

$(3+4)+6$ ___

$3\times(4\times 6)$ ___

$(3\times 4)\times 6$ ___

$(3+5)\times(6+4)$ ___

$(9+19)\times(42+8)$ ___

$(18+10)+5$ ___

$(16\times 13)+49$ ___

$49+(16\times 13)$ ___

$3\times((5\times 6)+4)$ ___

$(3\times(5\times 6))+4$ ___

$((2+3)\times(4+6))+(2\times 5)$ ___

$1+(2\times(3+(4\times(5+6))))$ ___

$((((1+2)\times 3)+4)\times 5)+6$ ___

1.2 Check your answers to Exercise 1.1 and repeat each one which is incorrect, filling in the steps of the evaluation in the manner shown in the text. For example, the last exercise would appear as follows:

```
    ((((1+2)×3)+4)×5)+6
    (((  3  ×3)+4)×5)+6
    ((      9  +4)×5)+6
    (          13 ×5)+6
                 65 +6
71
```

1.3 Enter numbers in the underscored positions such that each expression gives the indicated result:

| | |
|---|---|
| $(3+\_\_\_)\times 6$ | 42 |
| $3+(\_\_\_\times 6)$ | 27 |
| $(7+\_\_\_)\times 3$ | 30 |
| $(7+3)\times\_\_\_$ | 30 |
| $(42+\_\_\_)\times 4$ | 200 |
| $(\_\_\_+6)+4$ | 17 |
| $(2\times\_\_\_)+19$ | 49 |
| $\_\_\_+(45\times 6)$ | 274 |
| $(4+\_\_\_)\times(5+6)$ | 77 |
| $(3+(4\times(\_\_\_+2)))+7$ | 38 |
| $(2\times(((3+\_\_\_)+(2\times 2))+5))+3$ | 33 |

1.4 Check your answers for Exercise 1.3. For each one that is incorrect, show every step of the evaluation using the number that you entered in the underscored position.

1.5 Write an equivalent algebraic expression for each of the following sentences:

Quantity 7 plus 1 multiplied by 3.

17 added to the product of 6 and 2.

5 times the quantity $17\times6$.

Add the quantity $3+2$ to the product of 8 and 5.

The product of the quantities $6+10$ and $7+3$.

The sum of 4 and 14 added to the product of 3 and 13.

29 plus the product of 19 and 6.

Quantity $9+20$ added to the sum of 7 and 6.

Increase the quantity $8\times3$ by 7.

Add 15 to the sum of 14 and 8.

Multiply 6 times itself and then add 3.

Quantity $1+2+3$ times 8.

The product of $3+4$ and 8.

2 plus twice the quantity $9+5$.

Six more than the product of 2 and 8.

1.6 Write an equivalent English expression for each algebraic expression in Exercise 1.1.

1.7 Evaluate the following:

$2\times3+4$ ____

$2+3\times4$ ____

$1+2\times3+4\times5$ ____

$1+(2\times3)+4$ ____

$1+(2\times3)+4\times5$ ____

$(2+9+20)\times16$ ____

$14\times15\times13+6+20$ ____

$2\times10+10$ ____

$9\times(2+7)\times3$ ____

$23+7\times2+1$ ____

$1+(9\times11)+11\times1$ ____

$1+(2\times3+4)\times5+6$ ____

$1+(2\times3)+(4\times5)+6$ ____

1.8 For each wrong answer obtained in Exercise 1.7 take the given expression and modify it by inserting all of the parentheses implied by the rule to evaluate the rightmost function first. Then evaluate the resulting expression. For example:

$1+(2\times3)+4\times5$
$1+((2\times3)+(4\times5))$

27

1.9 Enter a number in each of the underscored positions such that the expression gives the indicated result:

2+____×3+5
162

2+(____×3)+5
67

2×(3+____)×5+3
144

2×(____+3)×5+3
144

10+6×4+____×2
130

10+(6×4)+____×2
130

10×25+____+45
800

____×9×3×1×7
9072

(____+40+10)×2
118

10+17+____×17×5
197

43+9×6+____
160

1.10 For each wrong answer obtained in Exercise 1.9, fill into the given expression your answer and all of the implied parentheses and then evaluate the resulting expression.

1.11 Using as few parentheses as possible, write algebraic expressions for each of the English expressions of Exercise 1.5.

1.12 Write equivalent English expressions for each of the expressions of Exercise 1.7.

1.13 Evaluate the following:

*A*←2
*B*←3

*A*+*B* ____

*A*×*B* ____

*A*+3 ____

4×*B*+8×*A* ____

(10+*B*)×*A* ____

*P*←9
*B*←2

*A*+*P*×3 ____

(*B*+*B*)×*P* ____

*A*+*B*+*B*+*B* ____

*A*+(3×*B*) ____

*A*+3×*B* ____

*A*+(*P*+7) ____

*SPEED*←60
*TIME*←5
*DISTANCE*←*SPEED*×*TIME*

*DISTANCE* ____

*SPEED*×7 ____

*SPEED*←40

*SPEED*×*TIME* ____

3×(4×*A*) ____

(4×*A*)×3 ____

(*A*×4)×3 ____

*A*×4×3 ____

*CAT*←1
*KITTENS*←4
*TOTAL*←*CAT*+*KITTENS*

```
        TOTAL
____

        NEWKITTENS←KITTENS×5
        TOTAL←TOTAL+NEWKITTENS

        TOTAL
____
        2×TOTAL+(4×7)
____
        (5+(CAT×TIME)+3)×3
____
        CAT×CAT+5
____
        CAT
____
        17+(17+TOTAL)×2
____

        T←4
        V←7

        V×(T+3)
____
        (T+3)×V
____
        (T×V)+(3×V)
____
        (V×T)+(V×3)
____
        V×T+V×3
____

        DO←3

        DO+6×7
____
        3+DO×4+5
____
        DO
____

        X←3

        X×X
____

        X←5

        X×X
____
```

1.14 For each wrong answer in Exercise 1.13, repeat the work showing every step of the evaluation.

1.15 Fill in the underscored positions so that the expressions give the indicated result:

```
        WIDTH←9

        (____+WIDTH)×3
93
        8+(____×WIDTH)
44

        LEN←____

        WIDTH+LEN
13
        (LEN×3)+(WIDTH+____)
22
        10+LEN×____
18

        HEIGHT←5

        20+HEIGHT+____
37

        VOLUME←LEN×WIDTH×HEIGHT

        ____×VOLUME
360
        (LEN+VOLUME)+____
190
        (3+4+LEN)×____
55
        (3+4)+(LEN×____)
55
```

1.16 For each wrong answer in Exercise 1.15, write in your answer and every step in the evaluation of the expression.

1.17 Translate each of the following sentences into a sequence of algebraic expressions:

The length of a playing field is 100 yards. Its width is 50 yards. The area is the length times the width.

A weightlifter has a steel bar weighing 20 lbs. He also has two weights, each weighing 50 lbs. The total weight that he will be lifting is the sum of the bar and the two weights.

A triangle has three sides. Side a is 3 inches long, side b is 4 inches, and side c is 5 inches long. The perimeter of the triangle is the sum of the lengths of the sides.

A nickel has a value of five cents. A dime is worth ten cents. A quarter is worth two dimes and one nickel.

An airplane is flying directly east with a heading of 90 degrees. He turns right 30 degrees. The new heading will be the sum of the old heading and the amount that the plane turned.

On a trip across the country, the Smiths travelled for six days, covering 500 miles each day. The total distance travelled is the daily mileage times the number of days in transit.

John weighed 100 lbs. He then ate three pieces of steak, each weighing 1 lb. His new weight is the sum of his old weight and all that he ate.

1.18 Make up "word problems" to correspond to each of the following groups of algebraic sentences:

```
      X←100
      Y←50

      X×Y
5000

      INCHES←7
      FEET←2
      YARDS←4

      (YARDS×36)+(FEET×12)+INCHES
175
```

1.19 Evaluate the following:

```
      +/9 7 19 19
____
      ×/4 2 1 6 3
____
      ×/20 5 7
____
      18+(×/20 3 1)
____
      (×/2 4)+39
____

      (+/10 20)×3
____
      +/43 7 19 21 28
____
      +/16 15 50 36
____
      +/30 4
____

      3+3+3
____
      3+3
____
      3
____
      +/3 3 3
____
      +/3 3
____
      +/3
____

      +/10 19
____
      +/30 7 45
____
      (+/3 4 1)×7
____
      +/7
____
      ×/8 3 7
____

      ABC←1 3 5
      DE←2 4 6 8 10
```

+/*ABC*

____

×/*DE*

____

×/*ABC*

____

*ABC*

____

+/*DE*

____

3++/*ABC*

____

2×2×2

____

2×2

____

2

____

×/2 2 2

____

×/2 2

____

×/2

____

×/19 19 5

____

+/9 10 1

____

7+×/3 5 7

____

×/3

____

(+/9 43 46 4)+7

____

×/13 5

____

*E*←2

*E*+×/*ABC*

____

(+/*DE*)×*E*

____

*E*←3+×/*ABC*

*E*+(+/*ABC*)+(+/*DE*)

____

(*E*×3)++/*ABC*

____

+/*ABC*++/*DE*

____

+/5

____

+/9 26 42 15

____

×/2 6 9 27 19

____

(×/12 49 45)×8

____

+/15 34 14

____

×/9

____

1.20 Use the <u>over</u> notation to write an equivalent algebraic expression for each of the following sentences:

Plus over 4 6 8 9.

Times over 2 4 6.

The sum over 20 15 4.

6 plus the product over 4 1 2.

2 plus the sum over 3 12 4 20.

The product of 3 and 7.

Ten times the product over 8 3.

Four plus 3 plus 7.

Three times the sum over 1 2 3 4 5 6.

Six times seven times one times three.

Quantity 4+3 times the sum over 20 17 4 7.

The sum of 3 4 and 5, all times the product over 2 8 3 4.

1.21 Write an equivalent English expression for each of the first 10 expressions in Exercise 1.19.

1.22 Evaluate the following:

```
      ⍳4
____
      +/⍳4
____
      ×/⍳4
____

      ⍳5
____
      +/⍳5
____
      ×/⍳5
____
      ⍳1
____
      +/⍳1
____

      N←3

      +/⍳N
____
      +/⍳N+1
____
      +/⍳N+2
____
      +/⍳2×N
____
```

1.23 Fill in the underscored position so that each of the expressions give the indicated results:

```
      ⍳____
1 2 3 4
      +/⍳____
10
      +/⍳____
15
      +/⍳____
55

      ×/⍳____
24
      ×/⍳3+____
720
      +/⍳4+____
78

      N←____

      ×/⍳N
120
      +/⍳____
1
      ×/⍳____
1
      ×/⍳____
3628800
```

1.24 Write an equivalent algebraic expression for each of the following sentences:

The first three integers.

Iota 5.

The integers to nine.

The sum of the first three integers.

Times over the integers to 4.

Plus over the integers to 7.

*Q* is assigned the value 4.

The integers to *Q*.

The one digit integers.

1.25 Write an equivalent English expression for each of the expressions of Exercise 1.22.

1.26 Evaluate the following:

```
      3 5 7 4+6 2 9 15
____
      4 3 2 1+1 2 3 4
____
      3 5 7 9+⍳4
____
      +/3 5 7 9+⍳4
____
      3 5 7 9+3
____
      3+3 5 7 9
____
```

```
      3+ι4
____
      5×ι4
____
      3+5×ι4
____
      (ι4)×(ι4)
____
      +/(ι4)×(ι4)
____

      N←3 5 7 9
      M←4

      N+M
____
      N+ιM
____
      N×N
____
      M+M×ιM
____

      +/3×ι6
____
      3×+/ι6
____
      3×4+ι5
____
      12+3×ι5
____
```

**1.27** Fill in the underscored positions so that each of the expressions give the indicated result (Note that each entry may be either a vector or a scalar):

```
      2 3 5 7+____
5 10 6 10
      2 3 5 7+____
6 7 9 11
      2 8 1 6×____
6 32 4 30
      2 8 1 6×____
10 40 5 30

      ____+ι4
6 7 8 9
      ____×ι4
20 40 60 80
      ____+ι____
8 9 10 11 12
      ____×ι____
7 14 21 28 35
      ____+____×ι____
8 13 18 23 28 33
      3×____+ι____
18 21 24 27 30 33
      ____×____+ι____
16 20 24 28 32
      ____+____×ι____
16 20 24 28 32
```

**1.28** Write an equivalent algebraic expression for each English expression:

The first five integers following 4.

Every third integer beginning with 3 and ending with 21.

Every third integer beginning with 7 and ending with 31.

**1.29** Evaluate the following:

```
      3ρ5
____
      +/3ρ5
____
      3×5
____
      ×/2ρ4
____

      ×/10ρ4
____
      (4ρ1)+2 3 5 7
____
      1+2 3 5 7
____
      (4ρ2)×ι4
____

      ×/9ρ10
____
      4××/3ρ7
____
      3++/3ρ7
____
      16 9 13 10++/4ρ4
____
```

1.30 Fill in the blanks so that the expressions give the printed result:

```
      +/4ρ___
12
      8ρ___
8 8 8 8 8 8 8 8
      +/8ρ___
64
      6ρ___
6 6 6 6 6 6
      ×/___ρ3
243

      ×/5ρ___
100000
      2ρ___
3 3
      +/___ρ10
80
      +/___ρ4
28
      +/___ρ6
60

      10ρ___
2 2 2 2 2 2 2 2 2 2
      ×/3ρ___
343
      ___ρ7
7 7 7 7 7 7 7 7 7
      8ρ___
5 5 5 5 5 5 5 5
      ___ρ2
2 2 2 2 2 2 2 2

      ×/9ρ___
134217728
      ___ρ5
5 5 5 5 5 5
      2ρ___
10 10
      ___ρ3
3 3 3 3 3 3 3 3 3 3
      ×/___ρ4
262144
      1ρ___
1
      7ρ___
3 3 3 3 3 3 3
      ___ρ7
7 7 7 7 7 7 7 7
      ×/___ρ1
1
      +/___ρ5
40

      ___ρ3
3 3 3 3
      ___ρ9
9 9 9
      ___ρ8
8 8 8 8 8 8
      +/___ρ5
5
```

1.31 Write an equivalent algebraic expression for each of the following sentences:

Three repetitions of 5.

5 repetitions of 3.

Plus over 6 repetitions of 4.

The product of 3 repetitions of 7.

Seven repetitions of six.

The sum of ten repetitions of four.

Times over vector 3 6 plus 2 repetitions of 5.

Vector 5 7 9 times 3 repetitions of 1.

4 repetitions of 7 plus 4 repetitions of 3.

3 times 6 repetitions of 5.

1.32 Evaluate the following:

```
N←2 3 5 7
M←8 7 6 5

M+N
______
M+N×M
______
(M+4)×M
______
(M+N)×M
______
(M+ι4)×N
______

((3×M)+(2×N))×2
______
+/3×M
______
3×+/M
______
+/M×N
______
M×+/N
______

(+/M)×N
______
(+/M)×+/N
______
×/M+N
______
(×/M)+N
______
(×/M)+×/N
______

ι+/N
______
+/ι+/N
______
+/ι+/ι3
______
+/ι+/3ρ2
______
×/ι×/2ρ3
______

2⌈3
______
M⌈N
______
⌈/M
______
⌈/N
______
⌈/M+N
______

⌈/M×N
______
(⌈/M)+⌈/N
______
(⌈/M)×N
______
+/M⌈N
______

×/M⌈N
______
4⌈N
______
+/4⌈N
______
×/4⌈N
______
```

# 2

2.1 Use Table 2.1 to evaluate the function "normal weight" for the following arguments:

59 63 69 60

2.2 We will use the term "two times" for a function whose result is twice the argument. Thus a table for this function for the arguments $\iota 4$ would appear as follows:

| Argument | Result |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |

a) Make a table for the "two times" function for the same set of arguments as used in Table 2.1.

b) Is the "two times" function a good approximation to the "normal weight" function of Table 2.1? Over what set of arguments do the two functions differ by not more than 2?

c) One could add a certain "correction" to each result of the "two times" function to obtain the exact normal weight. For example, for the argument 63 the value of "two times" is 126 and a correction of 4 is needed to give the actual normal weight of 130. Make a table to represent an appropriate "correction" function for the arguments from 60 to 70.

2.3 Evaluate the function represented by Table 2.2 for each of the following cases:

61 inches medium frame
58 inches large frame
63 inches small frame
65 inches all frames
68 inches small and large

2.4 Use the information in Table 2.2 to make tables to represent each of the following functions:

a) Normal weights for large frame and heights 60 to 66.

b) Normal weights for all frames and heights 66 to 70.

c) Normal weights for small frame and for even numbered heights from 58 to 68, that is, for heights $56+2\times\iota 6$.

d) Normal weights for height 67 and all frames.

2.5 a) Extend the table of Figure 2.3 to include arguments up to 12 (for both arguments).

b) Circle the result in the table which results from the expression $6\times 8$.

c) Underscore the result of the expression $8\times 6$.

d) Pick out all occurrences of the number 40 in your table and label each with a different letter of the alphabet. Then write these letters in a column and beside each write the expression (e.g., $5\times8$) which corresponds to that particular entry in the table.

e) Repeat part (d) for the number 24.

2.6 a) Construct an addition table for the arguments 1 to 12.

b) Label each occurrence of the result 9 in the table with a different letter. Then list the letters and show with each the expression which corresponds to that entry.

c) Repeat part (b) for the number 20.

2.7 Let $X$ denote the domain of the first argument of the multiplication table of Figure 2.3 (that is, $X\leftarrow\iota8$), and let $Y$ denote the domain of the second argument (that is, $Y\leftarrow\iota10$). Then the function represented by the third row of the body of Figure 2.3 can also be represented as $3\times Y$, and the function represented by the fourth column can be represented as $X\times4$. Use this scheme to write expressions which represent each of the functions represented by the following parts of the body of Table 2.3:

a) Row 2.
b) Column 10.
c) Row 5.
d) Column 5.

2.8 Make a table whose body consists of one column taken from the 8th column of the body of the multiplication table of Figure 2.3, and whose first column (that is, the arguments lying outside the body) is taken from the second column of the body of Figure 2.3. Call the function represented by this table $F$.

a) Evaluate the function $F$ for the arguments 4, 6, and 10.

b) What is the domain of $F$?

c) What is the range of $F$?

d) Write an expression (in the manner of Exercise 2.7) which represents $F$.

2.9 Repeat Exercise 2.8 using row 9 of the body of Figure 2.3 as the one-column body of the table, and row 3 as the arguments. If any of the arguments in part (a) do not lie in the domain of this function, indicate that they cannot be evaluated.

2.10 Repeat Exercise 2.9 using rows from the addition table constructed in Exercise 2.6 rather than Figure 2.3.

2.11 (Parts a-i) Answer the nine questions posed in Section 2.2.

2.12 Let $A\leftarrow1\ 2\ 3\ 4$ and $B\leftarrow1\ 2\ 3\ 4\ 5$. Then evaluate the following:

a) $A\circ.\times B$

b) $A\circ.+B$

c) $B\circ.\times A$

d) $B\circ.+A$

e) $B\circ.\times B$

f) $A\circ.+A$

2.13 Evaluate the following:

a) (⍳3)∘.×(⍳4)

b) (2×⍳5)∘.+⍳3

c) (2×⍳5)∘.+(2×⍳5)

d) 2×(⍳3)∘.×(⍳4)

e) 5+(⍳3)∘.×(⍳4)

f) 2×(⍳5)∘.+⍳5

2.14 a) Construct a function table according to the following specifications:

Left domain: ⍳4
Right domain: ⍳6
Body: (3×⍳4)∘.+⍳6
Name: *H*

b) Evaluate the following:

3 *H* 5

5 *H* 3

1 *H* 1

4 *H* (1 *H* 1)

4 *H* 1 *H* 1

2 *H* 1 *H* 2

2.15 a) Construct a function table according to the following specifications:

Left domain: 56+⍳16
Right domain: 1 2 3
Body: Same as Fig. 2.2
Name: *W*

b) Evaluate the following:

68 *W* 1

68 *W* 2

63 *W* 3

c) State clearly the relation between the function *W* and the function represented by Figure 2.2.

2.16 a) Construct the following function table:

Left domain: ⍳8
Right domain: ⍳8
Body: (⍳8)∘.+⍳8
Name: *PLUS*

b) Evaluate the following:

3 *PLUS* 5

4 *PLUS* 6

3×4 *PLUS* 6

2 *PLUS* 2×3

4×2 *PLUS* 2×3

(4×2) *PLUS* 2×3

2+3 *PLUS* 4

2 *PLUS* 3+4

2+3+4

2 *PLUS* 3 *PLUS* 4

2.17 Evaluate the following:

3⌈3

3⌊8

8⌈3

8⌊3

2×5⌈7

(5+2)⌈9

(5×2)⌈9

3⌈5⌊2

(3⌈5)⌊2

2.18 Evaluate the following:

```
10⌈8⌈6⌈14⌈7⌈9

⌈/10 8 6 14 7 9

⌊/10 8 6 14 7 9

A←10 8 6 14 7 9
B←17 4 13 2 19

⌈/B

⌊/B

(⌊/A)+⌊/B

⌊/A+B

(+/A)⌊+/B

(+/A)⌈+/B

⌊/A⌈B

⌈/A⌊B

+/A⌊B

A∘.⌈B

B∘.⌈A

B∘.⌊A
```

2.19 a) Evaluate the following:

```
+/3⍴4

4×3

+/5⍴3

5×3

+/10⍴10

10×10
```

b) Use multiplication to evaluate the following:

```
+/6⍴3

+/25⍴16

+/100⍴13

+/20⍴20

+/2000⍴512
```

2.20 Evaluate the following:

```
×/3⍴2

2*3

×/5⍴2

2*5

×/6⍴4

4*6

×/10⍴2

2*10

×/2⍴10

10*2
```

2.21 Evaluate the following:

```
A←2 3 4 5 6 7 8 9

1*A

2*A

3*A

4*A

A∘.*A
```

2.22 Evaluate the following:

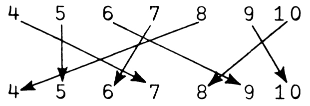
```
B←1 2 3 4 5 6

B*2

B*3

B*4
```

2.23 a) Let $F$ be the function represented by the following map:




Then evaluate the following:

$F$ 4

$F$ 6

$F$ 9

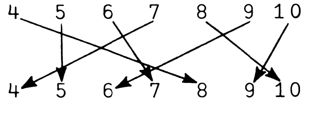
$F$ $F$ 6

$F$ $2\times3$

$2\times F$ 3

$F$ 4 5 6 7 8 9 10

b) Let $G$ be the function represented by the following map:




Then evaluate the following:

$G$ 4

$G$ 6

$G$ 7

$G$ $G$ 6

$G$ $2\times3$

$G$ 4 5 6 7 8 9 10

$F$ $G$ 4

$G$ $F$ 4

$F$ $G$ 6

$G$ $F$ 6

$F$ $G$ 4 5 6 7 8 9 10

$G$ $F$ 4 5 6 7 8 9 10

c) How are the functions $F$ and $G$ related?

d) Make maps of some other pair of functions $H$ and $K$ which are related in the same manner that $F$ and $G$ are.

e) Construct a function table to represent the function $F$.

f) Repeat part (e) for each of the functions $G$, $H$, and $K$.

2.24 Let $F$ and $G$ be the functions defined by maps in Exercise 2.23. Then if $X$ is any argument value, the expression $F\ G\ X$ means to apply the function $G$ to $X$ and then apply the function $F$ to the result.

a) Make maps to show the sequence of functions $F\ G\ X$.

b) Make a single map to show the overall result of the expression $F\ G\ X$.

c) State the overall effect of applying $F$ to the result of $G$.

d) Repeat parts (a-c) for the expression $G\ F\ X$.

e) Repeat parts (a-d) for the functions $H$ and $K$ of Exercise 2.23.

# 3

3.1 Evaluate the following:

```
      8-6
___
      13-6
___
      13-6 5 4 3 2 1
___
      6 7 8 9 10-5
___
      1 2 3 4 5+5
___
      8-⍳4
___
      +/8-⍳4
___

      M←8 12 7 11 43
      N←6 7 2 1 20

      M-N
___
      M+N
___
      (M-N)+N
___
      (M+N)-N
___
      M∘.+N
___

      2×+/⍳5
___
      5×6
___
      2×+/⍳6
___
      6×7
___
      2×+/⍳7
___
      7×8
___
```

```
      ⍳5
___
      6-⍳5
___
      +/⍳5
___
      +/6-⍳5
___
      2×+/⍳5
___
      (⍳5)+(6-⍳5)
___
      +/(⍳5)+(6-⍳5)
___

      P←7+⍳5
      P∘.-⍳5
___
```

3.2 Fill in the blanks so that the expressions will give the indicated results. Note that each entry may be either a scalar or a vector:

```
      8-___
5
      (8-___)+6
10
      (8+6)-___
10
      ___-2 3 4 5
6 9 1 8
      ___-⍳5
2 4 6 8
      +/8-⍳___
25

      M←2 3 5 7

      ___-M
8 7 14 2
      ___-M
6 5 3 1
```

3.3 In defining the over notation it was shown that +/14 10 8 7 2 means 14+10+8+7+2. Similarly, -/14 10 8 7 2 means 14-10-8-7-2, where the expression is evaluated from the right as usual. That is, -/14 10 8 7 2 is equivalent to 14-(10-(8-(7-2))), or 7. Use this fact to evaluate the following:

```
-/8 6 4 2
____
-/12 9 8 4 3
____
-/20 14 12 10 18 9
____
(20+12+18)-(14+10+9)
____

-/8 7 6 5 4 3 2 1
____
(8+6+4+2)-(7+5+3+1)
____
-/7 6 5 4 3 2 1
____
(7+5+3+1)-(6+4+2)
____
```

3.4 Make a map to represent each of the following:

```
7 8 9 10 11-5

2 3 4 5 6+5

10 11 12 13 14-8

2 3 4 5 6+8

((ι5)+6)-6
```

3.5 Evaluate the following:

```
5-8
____
5-ι8
____
1-ι8
____
8-ι8
____
0-ι8
____

S←ι8

S+S
____
S-S
____
S∘.-S
____

T←S+S

T∘.-T
____
T∘.-S
____
S∘.-T
____

-/10 8 6 4
____
-/1 2 3 4
____
-/ι5
____
-/ι6
____
-/ι7
____
-/ι8
____
```

3.6 Fill in the blanks so that the expressions will yield the indicated results:

```
      8-____
¯3
      8-____
3
      (8+____)-48
8
      3 4 5 12-____
0 2 ¯5 ¯8
      3 4 7 12-____
¯2 ¯1 2 7
      (ι5)-____
¯3 3 ¯3 3 ¯3
```

```
      +/3-ι___
0
      +/5-ι___
0
      +/7-ι___
0
      -/ι___
¯3
      -/ι___
¯4
      -/ι___
¯8
```

3.7 Make maps to represent the following:

```
      (ι5)-3

      (ι5)+¯3

      (ι7)-9

      (ι7)+¯9

      (ι7)+9

      (ι7)-¯9
```

3.8 Evaluate the following:

```
      (ι5)-3
___
      (ι5)+¯3
___
      (ι7)-¯9
___
      (ι7)+9
___

      N←0-ι6
      P←ι6

      N
___
      P
___
      P+N
___
      P-N
___
      N-P
___

      P∘.+N
___
      P∘.-N
___
      N∘.+P
___
      N∘.-P
___
```

3.9 Fill in the blanks so as to make the expressions yield the indicated results:

```
      ¯3 2 1 5-___
¯8 6 ¯4 18
      ¯4 1 ¯3 7 5+___
¯8 ¯5 ¯3 2 12

      S←¯8 ¯5 ¯3 2 12

      S-___
¯4 1 ¯3 7 5
      S+___
14 2 ¯8 3 7
      S+___
¯5 ¯2 0 5 15
      S+___
¯11 ¯8 ¯6 1 9
      S-___
0 0 0 0 0
      S+___
0 0 0 0 0

      +/0-ι___
¯10
      +/0-ι___
¯21
      -/ι___
3
      -/ι___
¯3
      -/ι___
4
      -/ι___
¯4
```

3.10 Write algebraic expressions for each of the following:

The integers from ¯8 to 8

The integers from ¯4 to 15

Every third integer from ¯12 to 12

Every second integer from ¯9 to 7

The positive integers to 6

The positive integers to 6 in descending order

The negative integers from ¯6 in ascending order (that is, running from ¯6 to ¯1)

The negative integers greater than ¯7 in descending order

# 4

4.1 a) Construct a subtraction table with a left domain of ι12 and a right domain of ι12.

b) Make a clear statement of each property you observe in the table of Part (a).

4.2 Two functions can be compared by comparing their tables. Try to arrange the tables produced in this exercise so that the comparisons among them required in Part (d) will be as convenient as possible. Let

```
A←ι3
B←ι4
S←A∘.-B
```

a) Evaluate the following:

```
⍉S

⌽S

⊖S

B∘.-A

0-B∘.-A
```

b) Without using any of the flipping functions ⍉, ⊖, or ⌽, write an expression to yield a result equivalent to ⍉*S*.

c) Evaluate the following:

```
⌽B

⌽S

A∘.-⌽B

⌽A

(⌽A)∘.-B

⊖S
```

d) State any relations you observe among the expressions of Part (c).

e) Write an expression using *S* (but not *A* or *B*) to yield a result equal to the result of the expression (⌽*A*)∘.-⌽*B*.

4.3 The following simple table *M* will be used to observe the behavior of the flipping functions:

```
      M←0 2∘.+1 2
      M
1 2
3 4
```

a) Evaluate the following expressions, arranging the tables for easy comparison:

```
      ⌽⊖M

      ⊖⌽M

      ⍉⌽M

      ⌽⍉M

      ⍉⊖M

      ⊖⍉M

      ⌽⍉⌽M
```

b) The expressions of Part (a) produce several different results although some pairs produce the same result. Using sequences of flipping functions as long as you like, how many __different__ results can you produce?

c) Can any sequence of flipping functions applied to *M* produce the result

```
      1 2
      4 3
```

d) Can you give a convincing argument to show that the different results you produced in Part (b) are __all__ that can be produced?

e) Write the shortest possible expressions you can find for each of the different results produced in Part (b). For example, the expression ⌽⍉*M* produces the result

```
      3 1
      4 2
```

and is therefore equivalent to rotating *M* clockwise by one position. Hence a re-application of the pair ⌽⍉ (that is, ⌽⍉⌽⍉*M*) will effect a second rotation to produce the result

```
      4 3
      2 1
```

However, this can also be produced by the shorter expression ⊖⌽*M*.

f) From the preceding parts of this exercise it should be clear that ⊖⌽*M* is __not__ equivalent to ⍉*M*. Nevertheless, for the subtraction table *S* it was obvious from the examples given in the text that ⊖⌽*S* is equivalent to ⍉*S*. What is there about the table *S* that makes this so?

4.4 Let

```
      A←3+⍳6
      B←2×⍳5
      M←A∘.-B
```

a) Evaluate the following:

```
      A[4]

      B[2]

      M[3;5]

      M[5;3]


      (⍉M)[3;5]

      (⍉M)[5;3]

      M[2;]

      M[;4]

      (M[3;])[5]
```

b) Evaluate the following:

```
A[2 4]

A[⍳3]

A[3+⍳3]

M[2 4;1 3 5]

M[2 4;]

M[;1 3]

A[3]

B[4]

A[3]+B[4]

(A∘.+B)[3;4]
```

4.5 Consider the addition table *B* given in the text. State any patterns you observe in the table. Where possible make your statements in both English and algebra. For example:

⍉*B* is equal to *B*.

0-⌽*B*[2;] is equal to *B*[2;].

*B*[*I*;] is equal to *B*[;*I*] for any value of *I*.

*B*[5;] is equal to 2+*B*[3;].

4.6 Repeat the work of Exercise 4.5 for the multiplication table *N* given in the text.

4.7 Quadrant 2 of the multiplication table *N* given in the text consists of the first seven rows and first seven columns of *N*. Hence Quadrant 2 is the table *Q*2 defined as follows:

```
Q2←N [⍳7;⍳7]
```

Quadrant 4 can be specified similarly:

```
Q4←N[8+⍳7;8+⍳7]
```

a) Write similar expressions to define the remaining quadrants *Q*1 and *Q*3.

b) State any relations you observe among the quadrants.

4.8 Repeat the work of Exercise 4.5 on the table *MAX* defined in the text.

4.9 Repeat the work of Exercise 4.5 on the table *MIN* defined in the text.

4.10 Evaluate the following and compare the results:

```
I←⍳6
J←0-I

I∘.⌈I

J∘.⌊J

I∘.⌊I

J∘.⌈J
```

4.11 a) Repeat Exercise 4.10 with *I*←(⍳13)-7.

b) Evaluate the following expressions and comment on the patterns in the table *T*:

```
K←⍳8
R←K∘.⌈K
T←R⌈⊖⌽R
```

4.12 Evaluate the following:

```
3=7

3=3

X←⍳7
Y←⌽X
```

```
X=Y

X≠Y

X∘.=X

X∘.≠X

X∘.=Y
```

4.13 Evaluate the following:

```
X←⍳7

X∘.>X

X∘.≥X

X∘.≤X

X∘.≤⌽X

⍉X∘.≥X
```

4.14 Evaluate the following:

```
I←(⍳11)-6
A←I∘.+I

4≤A

16≤A*2

S←I∘.-I
M←I∘.×I

4≤S

12≥M

144≥M*2
```

4.15 Evaluate the following:

```
X←8 4 3 5 7 6
Y←4 3 10 8 2 5

X≤Y

⌈/X≤Y

⌊/X≤Y

5≤X+Y

⌈/5≤X+Y

⌊/5≤X+Y

⌈/9≤X+Y

⌊/9≤X+Y

⌈/15≤X+Y

⌊/15≤X+Y
```

4.16 Evaluate the following:

```
A←(⍳6)∘.+⍳6

A=⍉A

⌊/A=⍉A

⌊/⌊/A=⍉A

S←(⍳6)∘.-⍳6

S=⍉S

⌊/⌊/S=⍉S

⌈/S=⍉S

⌊/⌈/S=⍉S

C←(⍳6)∘.≥⍳6

+/C

+/⍉C
```

# 5

5.1 Evaluate the following:

```
      4×8
___
      32÷4
___
      32÷8
___
      48÷8
___
      (32÷8)+(48÷8)
___
      (32+48)÷8
___

      S←6×⍳7

      S
___
      S÷2
___
      S÷3
___
      S÷6
___
      S÷1
___

      S∘.÷1 2 3 6
___
      (-S)÷2
___
      ¯3 ¯6 ¯9 ¯12 × 2
___

      T←S-24

      T
___
      T∘.÷1 2 3 6
___
```

```
      P←¯8 12 ¯10 21
      Q←16 15 35 49
      R←2 3 5 7

      P÷R
___
      Q÷R
___
      (P÷R)+(Q÷R)
___
      (P+Q)÷R
___
      (P-Q)÷R
___
```

5.2 Fill in each underscored position giving either the result of evaluating the expression or a value such that the expression will yield the indicated result:

```
      ___×3
24
      24÷3
___
      ___×15
300
      300÷15
___

      ___÷20
25
      25×20
___
      ___÷7
32
      32×7
___

      (25÷5)⌈(35÷5)
___
```

```
      (25⌈35)÷5
____
      (28+___)÷5
40
      ___÷5
¯40
      (28+___)÷5
¯40
      (¯28+___)÷5
40
      (¯28+___)÷5
¯40
      22 21 32÷___
2 7 16
      22 21 32÷___
22 21 32
```

5.3 Make maps to represent each of the following, where `S←ι6` and `N←¯4+ι7` and `M←4×S`:

```
S×5
S×5 followed by (S×5)÷5
N×2 followed by (N×2)÷2
M÷4 followed by (M÷4)×4
```

5.4 Evaluate the following using the method of guessing, first obtaining two guesses which "bracket" the result (that is, one is too high and the other is too low), and then closing in on the result by successive guesses which lie between the guesses which bracket the result most closely. Make your guesses as good as possible to shorten the work, but show all of your work:

```
256÷8
378÷7
4096÷16
5040÷42
40320÷105

362880÷144
362880÷27
362880÷48
362880÷36
```

5.5 Evaluate the following, using the method of guessing at a quotient, subtracting from the dividend the product of this guess with the divisor, making a guess at the quotient of the new remainder divided by the divisor, and so on. Show all of your work.

```
40548÷124
51324÷78
971203÷257
2511930÷1095

5764896÷2164
1505625÷1375
751424÷3184
```

5.6 Repeat the examples of Exercise 5.5 using the method of long division.

5.7 Fill in the blanks in the following, using long division where necessary.

```
      241724÷178
___
      ___×314
853452
      314×___
1174046
      (___+15)×624
144144
      (___-48)×176
457248

      (___÷3)÷167
416331
      2578647÷(167×3)
___
      (268000÷4)÷250
___
      268000÷(250×4)
___
      (238750×5)÷50
___
      23870÷(50÷5)
___

      1728÷12
___
      1728÷12*2
___
      1728÷12*3
___
```

5.8 Make maps to represent each of the following, where S←¯4+⍳9:

```
      S÷4 followed by (S÷4)×4

      S÷3 followed by (S÷3)×6

      S÷6 followed by (S÷6)×3
```

5.9 State the values of the divisor, dividend, and quotient for the following:

```
      8÷4
      10÷2
      196÷14
      2048÷64
      1728÷144
```

5.10 State the values of the numerator and denominator for each of the expressions of Exercise 5.9.

5.11 Give appropriate names for the following fractions:

```
      1÷2
      1÷3
      2÷5
      7÷5

      2÷6
      3÷6
      4÷6
      6÷6
      ¯7÷12
```

5.12 Under each expression below enter a simpler equivalent expression of the form *A*÷*B* (where *A* and *B* are integers), as shown by example in the first four lines:

```
      (2÷8)+(5÷8)
7÷8
      (7÷3)+(8÷3)
15÷3

      (10÷7)+(4÷7)

      (¯6÷13)+(32÷13)

      (32÷13)-(6÷13)

      (42÷15)+(¯42÷15)

      (26÷3)-(¯22÷3)

      (38-47)÷19

      (25+14)÷7

      (25+9)÷(4+5)

      (19+¯38)÷(7+8)

      (3÷9)-(25÷9)+(¯20÷9)
```

(10+27)÷(4-3)

(¯32÷12)-(¯32÷12)

(¯1÷18)-(¯19÷18)-(6÷18)

(2÷11)+(2÷11)+(2÷11)

(3×2)÷11

5.13 Review each of the results obtained in the preceding exercise and add a third line giving an equivalent integer if there is such an integer. For example:

(7÷3)+(8÷3)
15÷3
5

5.14 Fill in the underscored expressions with integers such that the indicated equivalences will hold:

(5÷13)+(____÷13)
19÷13

(5÷13)+(____÷13)
2

(16÷31)+(____÷31)
8÷31

(____÷17)+(21÷17)
¯2

(31÷99)-(____÷99)
22÷99

(64÷19)-(____÷19)
64÷19

(29÷____)+(19÷____)
4

5.15 Under each expression enter a simpler equivalent expression of the form integer ÷ integer:

(2÷3)×(5÷7)

(3÷5)×(5÷3)

(¯10÷17)×(51÷2)

(¯2÷3)×(¯2÷3)

(4÷7)×(7÷9)+(15÷9)

(13÷8)×(14÷6)-(¯17÷6)

((13÷8)×(11÷3))+((7÷12)×(5÷2))

((3÷4)+(10÷4))×(35÷15)-(19÷15)

(¯2÷8)×(¯5÷3)

(0÷4)×(¯15÷3)

(¯7÷5)×(5÷5)

(3÷4)×(12÷12)

5.16 Review each result obtained in the preceding exercise and give an equivalent integer where possible.

5.17 Fill in the underscored positions appropriately:

(3÷5)×(____÷12)
18÷60

(17÷8)×(2÷____)
34÷120

(15÷____)×(____÷20)
120÷80

(17÷24)×(____÷____)
85÷96

(5÷____)×(6÷____)
30÷54

(5÷3)×(____÷____)
4

(5÷3)×(____÷____)
1

(17÷23)×(____÷____)
1

(____÷____)×(39÷41)
1

5.18 Under each expression enter an equivalent expression of the form integer ÷ integer:

```
(2÷3)×(2÷2)

(2÷3)×(3÷3)

(3÷4)×(5÷5)


(7÷9)+(2÷3)×(3÷3)

(7÷9)+(2÷3)×1

(7÷9)+(2÷3)


((8÷4)×(5÷5))+(¯13÷20)

((3÷4)×(5÷5))+((3÷5)×(4÷4))

((2÷3)×(2÷2))+((1÷2)×(3÷3))
```

5.19 For each expression write an equivalent expression of the form integer ÷ integer:

```
3×(4÷5)

4×(3÷5)

5×(3÷5)


(7÷9)×11

3×(7÷9)×3

3×7÷9×3

(7÷9)×(3÷3)


5×14÷13×2

1×2÷3×4

1×(2÷3)×4

4×3÷2×1
```

5.20 As shown in the first example, write equivalent expressions of the form ÷/*V* where *V* is a vector whose two elements are integers:

```
(÷/3 5)×(÷/2 3)
÷/6 15


(÷/16 28)×(÷/10 20)

(16÷28)×(10÷20)

(10÷7)×(¯12÷3)


(÷/23 4)×(÷/4 23)

(÷/12 25)×(÷/4 4)

(3÷12)+(5÷12)

(÷/3 12)+(÷/5 12)

(÷/15 28)+(÷/¯1 28)

(÷/17 29)-(÷/¯32 29)


(÷/2 5)×(÷/3 7)

÷/2 5×3 7

2×÷/4 5

÷/2×4 5

5×÷/2 3×4 7
```

5.21 For each expression write an equivalent expression which involves not more than two integers:

```
(2÷7)+(4÷5)

(3÷5)+(4÷6)

(12÷24)+(¯3÷17)

(12÷24)-(3÷17)

(12÷24)-(¯3÷17)
```

```
(2÷5)+(3÷10)

(÷/2 5)+(÷/3 10)

(÷/5 2)+(÷/10 3)

2×(÷/5 2)-(÷/¯10 3)

2 7×(÷/5 2)+(÷/3 11)


3 3×(÷/5 7)-(÷/¯11 6)

(1÷2)+(3÷4)+(5÷6)

(1÷2)+(2÷3)+(3÷4)

(÷/⍳2)+(÷/1+⍳2)+(÷/2+⍳2)
```

5.22 Under each expression write a series of equivalent expressions showing the steps in simplifying to a final expression of the form *X÷Y*:

```
A←4 7
B←2 5

(÷/A)+(÷/B)

(÷/A)-(÷B)

(÷/B)-(÷/A)

(÷/B)+(÷/A)

(÷/A)+(÷/A)

G←10 9

(÷/B)-(÷/G)

(÷/A)+(÷/G)-(÷/G)

(÷/G)-(÷/B)

W←(÷/A)+(÷/G)

(÷/W)-(÷/A)

(÷/W)+(2×(÷/A))

(÷/B)+(÷/W)

(÷/B)×(÷/A)


(÷/W)×(÷/G)

(÷/W)×(÷/W)

(÷/G)×(÷/B)

(÷/W)+(÷/B)
```

5.23 For each expression write a simpler equivalent expression involving at most two integers:

```
(9÷2)÷(4÷3)

(7÷3)÷(4÷9)

(¯7÷3)÷(4÷9)

3÷(4÷9)

5÷(5÷6)

A←3 4
B←5 6

(÷/A)÷(÷/B)

(÷/B)÷(÷/A)

(÷/A)×(÷/B)÷(÷/A)


(÷/78 23)÷(÷/45 3)

(4÷7)÷(÷/3 1)

(4÷7)÷3

(7÷8)÷2
```

**5.24 Write the following rational numbers as decimal fractions:**

```
5÷10

2÷10

8÷1

34÷100

34÷1000

34÷10

7÷10000

234÷10

÷/234 100

234÷1000

45÷10

÷/294 10000

38÷10

50÷10

÷/23 100

÷/¯8 1000

÷/¯567 1

10000÷100

4567÷100

28345÷1000

79÷1000

÷/¯78 1000

÷/293847 10

29÷1

9287654÷100000

9÷100000

23÷100

36887÷10
```

**5.25 Write decimal fractions equivalent to the following:**

```
(÷/14 10)×(÷/7 100)

(÷/14 10)÷(÷/100 7)

(÷/24 100)×(÷/74 10)

(÷/14 100)+(÷/27 100)

(÷/64 100)+(÷/136 100)

(÷/164 100)+(÷/135 10)

(÷/13.6 10)+(÷/14.82 10)

(÷/15.66 10)×(÷/256.4 100)
```

**5.26 Evaluate the following showing each rational result as a decimal fraction:**

```
V←6 27 135
E←10*⍳4

V∘.÷E

F←(10*⍳7)÷1000

V∘.÷F
```

**5.27 Evaluate the following:**

```
34.3+6.3

2.5+5.6

19.4-3.2

38.6-¯10.3

(÷/48 10)+4.6

6.00+3.87
```

```
4.7300+9.4529+98.0000
¯7.50+68.90-¯548.21
5.78-2.40
¯67.8+3.6
866.00+(4÷100)
¯13.6-¯7.2
```

**5.28** Evaluate the following:

```
5.3+8.27
8.6+5.14+1.26
870.3458+¯78.2
(÷/34 10)+21.7-¯44.4
45.23+(÷/37 10)

(÷/56 100)+(4÷10)
5.6-(45÷10)+¯4.12
19.5-279.69
58.3-23.45
¯67.8+¯692.5678

(÷/¯93 1000)+2.45
(÷/98 100)+(12÷1)-(÷/98 10)
36.5-¯578.4
77.777-66.66
¯46.9-26.879
```

**5.29** Obtain decimal fraction equivalents for the following:

```
3÷4
1728÷25
1728÷16
153÷12
2 3÷5 25
3÷5 25

(⍳8)÷8
(⍳16)÷16
(⍳32)÷32
(⍳25)÷25
(⍳25)÷4

1÷2*⍳6
1÷5*⍳6
1÷10*⍳6
1-(⍳8)÷8
1-(⍳32)÷32
```

**5.30** Obtain the best 3-place decimal fraction approximation to the following:

```
1÷3
2÷3
(⍳9)÷9
(⍳32)÷32
(⍳10)∘.÷(⍳10)
(65÷24)÷(12÷44)
```

(71÷3)÷(7÷8)

(46÷9)÷(11÷19)

(32÷21)÷(12÷10)

(24÷28)÷16

(4÷9)÷(21÷8)÷(32÷6)

(7÷12)÷(25÷71)

(8÷1)×(6÷37)

(8÷13)÷(20÷9)

(7÷14)÷(31÷6)

(66÷2)÷(2÷3)

(9÷16)÷(6÷6)

(7÷2)÷(8÷3)÷(9÷4)

5.31 Evaluate the following:

2.41×1.48

3.27×16.4

1.287×14.321

234.56×12.34

2.4×3.5×4.6×5.7

13.287×4.8+5.6

1.125×.32

5.32 Obtain the best 2-place decimal approximation to each of the expressions of the preceding exercise.

5.33 Find the best 3-place approximation to each of the expressions of Exercise 5.31 but with each multiplication replaced by division.

5.34 Write each of the results of Exercise 5.31 in exponential notation.

5.35 Write each of the results of Exercise 5.33 in exponential notation with the value 3 for the integer following the *E*.

5.36 Obtain the best 3-place approximation to the following:

2÷3

2÷¯3

¯2÷3

¯2÷¯3

# 6

6.1 Evaluate the following:

```
A←2 3 5 7
B←4 1 2
C←9 8

A,B

B,A

(A,B),C

A,(B,C)

(-⌽⍳4),⍳4
```

6.2 Let *D* be the 8-by-8 division table shown in the text.

a) Evaluate the following:

```
D=1

D=1÷2

D=1÷3
```

b) Examine the results of Part (a) and state the pattern produced by expressions of the form *D=R*, where *R* is any value which occurs more than once in *D*. (if necessary evaluate further cases, possibly extending the table *D* itself)

6.3 a) Give expressions of the form used in Exercise 4.7 (for the multiplication table *N*) to define four suitable quadrants of the division table *J*∘.÷*K* given in Section 6.3.

b) State any relations observed among the quadrants.

6.4 a) Evaluate the following:

```
A←6 7 8 9 10 11
B←7 8 9 10 11 12
C←9 10 11 12 13 14
D←10 11 12 13 14 15

A÷B

C÷D

T←(A÷B)∘.≤(C÷D)
```

b) Use the table *T* to determine which is the larger of each of the following pairs of rationals:

8÷9 and 9÷10
9÷10 and 10÷11

c) Without using division write an expression which will yield a table identical with *T*. Evaluate the expression and compare the result with *T*.

6.5 Evaluate the following:

```
2 3∘.*1+⍳10

2*¯4+⍳12

3*¯4+⍳12

2 3∘.*¯4+⍳12

2 3 4 5 6∘.*¯4+⍳7
```

6.6 a) Evaluate the following to five decimal places:

```
A←⍳5
B←0-A

2*A

2*B

(2*A)×2*B

(2*⍳8)×2*0-⍳8

+/(2*⍳100)×2*0-⍳100
```

b) Evaluate the following to five decimal places:

```
A←⍳4

3*A

3*0-A

(3*A)×(3*0-A)
```

c) Evaluate the following to five decimal places:

```
A←⍳4

5*A

1÷5*A

5*0-A

(1÷5*A)=5*0-A
```

6.7 Evaluate the following:

```
A←⍳5

10*A

10*0-A

¯10*A

¯10*0-A

20*A

¯20*A

0*A

0*0-A
```

6.8 Evaluate the following:

```
A←(⍳6)÷2

9*A

B←0-A

9*B

49*A

49*B
```

6.9 a) Determine a number *A* which when multiplied by itself yields 10 (correct to three decimal places).

b) Use the result of Part (a) to evaluate the following:

```
10*(⍳6)÷2

10*0-(⍳6)÷2
```

6.10 Evaluate the following:

```
A←⍳6

3*A÷3

3*A÷4

3*A÷5

3*A÷6

5*A÷6
```

# 7

**7.1** Evaluate the following:

```
3|17
3|17 593 18 42
9|⍳9
9|+/⍳9
+/3|⍳3

+/4|⍳4
+/12|⍳12
(8+7)|6 1 654 42 752 9104
(5|77 5 24 2750 4 2)-660
3|+/6 8

(3+9≤6)|726 10 9 234 3064 36
2|-/2 4 9 653 1504 7
(4⌈6)|6 2522 5 5
5|5 1312 9 1 162×5 9 932 7 3
7|7 4394⌊955 8513

(9|7418 20 887)+200 10 866
(3×6)|7 1 1 8 4+7 21 82 4 10
|/3 9 1365+10 258 3
7|60 3
(6|2)×755
6|4≥3
6|3216 5 5172
|/4 19
|/7 37
```

**7.2** Evaluate the following:

```
1 2 3 ∘.| 4 5 6 7
2 4 6 ∘.| 6 12 18 24 72
1 4 ∘.| 10 4 520 831
4 7 8 10 ∘.| 920 6540 42
7≥10 50 3 1 10 ∘.| 3 360 2 5

8 8 10 89 3 ∘.| 9410 6 8
10 3 ∘.| 220 7148 14 910 7
10 56 2 ∘.| 4 388
7| 1 17 1 4 26 ∘.+ 3 1
10 7 5 9 7 ∘.| 3 680 9090 26

22 5 4 6 89 ∘.| 8 48 67 7 2
7-8 9 10 ∘.| 8 320
2 1 2 ∘.| 10 4 3 6922
(1+2≥1 6 3 4) ∘.| 3 7
```

```
5 4 3 1 ∘.| 529 4 6|2 486 9

8 10 7 3 8 ∘.| 10 25 85 69 5

7 7 8 6 ∘.| 4365 7 585

6|6 7 34 1 ∘.+ 3 1 5 458


70 5 3 ∘.|36 84 10 26 2516

5 8 ∘.| 69 4 9

(3×10 5) ∘.| 24 66 2 8

6 2 9 4 5 ∘.| 3 2 373+8 1 45
```

7.3 Evaluate the following:

```
3|¯10

(3+7)|9 5270 ¯1 ¯7 ¯4160

10|¯727 ¯7 ¯9 ¯3

(4+2)|¯61

6|+/¯465 0


1|¯8

6|8416⌊0 0 4 6 5

16|×/1 5 3

66|4998 1476⌊¯10

(10×5)|2⌊9 25 48


(26×10)|350 46 9 ¯94 6517

|/¯5+7023 99 ¯10

8|¯6 2 ¯451 990 ¯216

(5+5≥9)|-6164 1 ¯1

1|¯3 1×¯3899

(3|3 ¯8198)×¯9 2

7|=/¯7 ¯1

(4⌈3≤6)|¯9 9 6 0≥44 3 38 40

6|¯9402 3216
```

7.4 Make a table of the results of the expression `(⍳9)∘.|¯10+⍳19`. Do you notice any patterns in the table? Are they similar to the patterns in Table 7.1? Draw circles around all the `0`'s in the table. Connect groups of these circles by straight lines. Does it seem that one half of the table is the mirror image of the other half with respect to these lines?

7.5 Evaluate the following:

```
0=3|⍳16

0=5|⍳25

M←(10×0,⍳9)∘.+0,⍳9
4|M

9|M

7|M
```

7.6 Make the table `0=(⍳10)∘.|⍳10`. Circle the positions of all the `1`'s in the table. Why are there no `1`'s in half of the table? What is the significance of the line of `1`'s that divides the table in half?

7.7 In the table of the preceding exercise, the number `3` will be seen to have exactly two divisors (`1` and `3`). Find all the other numbers in the table which have exactly two divisors. Find four more numbers not in the table which have this property.

7.8 Make the table 0=(⍳10)∘.|¯11+⍳21. Note all of the interesting properties of the table that you can observe; for example, is the left half a mirror image of the right half? Where does the split occur? Is ¯8 divisible by the same numbers as 8?

7.9 Which of the following numbers is divisible by 3:

12 45 34 87 ¯10 5 76543 76 ¯567 9876543 39 ¯149 9378 345 83 86 ¯237 873 3482 ¯93754

Add up the digits of each number. Are these sums divisible by 3? Can you find a rule that will tell quickly whether a number is divisible by 3 or not? Can you find a relationship between the 3-residue of the number and the residue of the sum of its digits? Does this relationship hold for integers other than 3?

7.10 Which of the following numbers is divisible by 5?

56 25 90 1234 1000 ¯595 98765 234 ¯3591 63 55 80 ¯390 48 240

Is there any relationship between the 5-residue of the number and the 5-residue of its final digit?

7.11 Which of the following numbers is divisible by 2?

8 24 86 ¯456 9870 34592 ¯237 162 1000 645 343 926 ¯427 1445 92

Is there any relationship between the 2-residue of a number and the 2-residue of its final digit?

7.12 Write down in your own words a definition for the | function. According to your definition, what would the result of 0|*N* be, where *N* is any integer?

Now suppose you defined *A*|*B* as the repeated subtraction of *A* from *B* until a result is obtained that is 0 or larger but also less than *A*. Will this definition produce the same results as the definition introduced in the text? Using this new definition, 0|*B* would be a never ending process. Would it seem reasonable to let 0|*B* have the result *B*?

7.13 Evaluate the expression (⍳*N*)|*N* for each of the following values of *N*:

9 12 15 17 24 32 36

7.14 Use the results of the preceding exercise to determine all of the factors of each of the numbers 9,12, etc., listed in that exercise.

7.15 For each list of factors obtained in the preceding exercise write the list of corresponding factor pairs. For example, the factors of 6 are 1 2 3 6 and the corresponding factors are 6 3 2 1.

7.16 From your answers to the preceding exercise, does it seem reasonable that every number has an even number of factors? Can you find any numbers that have an odd number of factors? If a number has an odd number of factors, what are its factor pairs?

7.17 Evaluate the following:

```
1 0 1 0 1/3 5 7 9 11

0 1 0 1 0/3 5 7 9 11


X←12 17 ¯4 5 ¯3 0 ¯4 0

1 1 1 1 0 0 0 0/X

(X>0)/X

(X≥0)/X

(0=2|X)/X

(0=3|X)/X


((0=2|X)⌈(0=3|X))/X

((0=2|X)⌊(0=3|X))/X

((0=2|X)⌊(0≠3|X))/X

(0=5|⍳25)/⍳25

(1=5|⍳25)/⍳25


(2<5|⍳25)/⍳25

+/X∘.=X

(1=+/X∘.=X)/X

(1≠+/X∘.=X)/X
```

7.18 Write expressions which will select from the positive integers up to *N* those numbers satisfying the stated properties. For example, the expression `(0=4|⍳N)/⍳N` would be appropriate for the property "all integers up to *N* which are divisible by 4".

a) All integers up to *N* which are divisible by either 3 or 5

b) All integers up to *N* which are divisible by both 3 and 5

c) All integers up to *N* which are divisible by 15

d) All integers up to *N* which are greater than *M*

e) All integers up to *N* which are greater than *M* and divisible by 5

f) All integers up to *N* which are divisible by every element of the vector *V*

g) All integers up to *N* which are divisible by exactly *K* elements of the vector *V*

7.19 Use the expression `(2=+/⍉0=(⍳N)∘.|⍳N)/⍳N` to determine all of the prime numbers up to 20. Show each step of the calculation.

7.20 Evaluate the following:

```
P←(2=+/⍉0=(⍳12)∘.|⍳12)/⍳12
P⋆2 0 2 0 1

×/P⋆2 0 2 0 1


×/P⋆0 0 0 0 0

×/P⋆1 0 0 0 0

×/P⋆0 1 0 0 0

×/P⋆2 0 0 0 0

×/P⋆0 0 1 0 0

×/P⋆1 1 0 0 0
```

7.21 The expressions of the preceding exercise were all of the form `×/P⋆E`, and the last six of them yielded the first six positive integers. Determine further values of *E* to continue the process for integers 7, 8, 9, etc. What is the first integer impossible to represent in this way?

7.22 Take the first integer which cannot be represented in the form $\times/2\ 3\ 5\ 7\ 11{*}E$ and append it (it is a prime number) to the list $P$ and then continue the process of Exercise 7.21 for a few more integers. Can every integer be represented as $\times/P{*}E$ where $P$ is a vector of prime numbers?

7.23 a) If $P$ is a vector of primes and if $M\leftarrow\times/P{*}E$ and $N\leftarrow\times/P{*}F$ and $G\leftarrow\times/P{*}E\lfloor F$, then $G$ is a divisor of both $M$ and $N$. Choose a number of different values of $E$ and $F$ and verify that this is so for the cases chosen.

b) Explain why $G$ is a divisor of both $M$ and $N$.

c) Is it possible to find a number larger than $G$ which divides both $M$ and $N$? Why?

7.24 a) If $P$, $M$, and $N$ are as defined in the preceding exercise, and if $L\leftarrow\times/P{*}E\lceil F$, then both $M$ and $N$ divide $L$. Verify this for a few values of $E$ and $F$.

b) Explain why $M$ and $N$ divide $L$.

c) Is it possible to find a number smaller than $L$ which is divisible by both $M$ and $N$? Why?

# 8

8.1 Evaluate the following:

```
!3

×/⍳3

!4

×/⍳4

!⍳10


(!5)÷(!4)

(!6)÷(!5)

(!1+⍳10)÷(!⍳10)

(1+⍳10)×(!⍳10)

(!⍳10)÷⍳10

1,!⍳9
```

8.2 Comparison of the last two results of Exercise 8.1 suggests a definition for the value of `!0`. What is the value? Would its adoption agree with the obvious requirement that `!N+1` is equal to `(N+1)×!N`? What value would the same line of reasoning give for `!¯1`?

8.3 Evaluate the following:

```
-⍳6


X←2 ¯5 3 ¯7 4

-X

X-3

X+¯3

X+-X

X--X

X×-X

X⌈-X

X⌊-X
```

8.4 Evaluate the following correct to 3 decimal places:

```
÷4

÷5

÷6

÷⍳8

-÷⍳8


÷-⍳8

÷!⍳5

+/÷!⍳5

÷2*⍳5

+/÷2*⍳5
```

8.5 Evaluate the expression `+/÷2*⍳N` for the first few positive values of `N`. What integer do these results seem to be approaching? Can you choose a positive value of `N` large enough so that `+/÷2*⍳N` is larger than `1`?

8.6 a) Evaluate the following:

```
| 3 ¯4 7 ¯9 ¯10


X←3 ¯4 7 ¯9 ¯10
```

```
|X

|-X

-|X

X÷|X

+/|X

|+/X

X=|X

(X=|X)/X

(X≠|X)/X
```

b) Evaluate the following:

```
P←7.2 ¯3.4 8.1 ¯6

|P

P⌈-P
```

c) What is the relation between the expressions |*P* and *P*⌈-*P* appearing in Part (b)? Would this relation remain true for any value assigned to *P*?

8.7 Evaluate the following:

```
⌊3.5 ¯2.6 2 ¯4.9

⌈3.5 ¯2.6 2 ¯4.9

⌊(ι10)÷2

⌈(ι10)÷2

⌊(ι10)÷3

⌈(ι10)÷3

X←1.8 ¯2.7 ¯6 4.9 7

X=⌊X

(X=⌊X)/X

(X≠⌊X)/X

N←ι12

⌊N÷3

(N-3|N)÷3

⌊N÷5

(N-5|N)÷5
```

8.8 Evaluate the following:

```
~1 1 0 1 0 1

~~1 1 0 1 0 1

X←¯2 3 ¯5 7 11

X>3

~X>3

X≤3

~0=5|ι12

0≠5|ι12
```

8.9 Evaluate the following and compare their results:

```
L←0 1

L∘.⌊L

~(~L)∘.⌈(~L)

L∘.⌈L

~(~L)∘.⌊(~L)

L∘.≠L
```

```
~(~L)∘.=(~L)

L∘.<L

~(~L)∘.≤(~L)
```

8.10 If *L* is any logical vector (i.e., each of its elements is either 0 or 1) of any dimension, then the expressions ⌊/*L* and ~⌈/~*L* yield the same result.

a) Verify this for a number of values of *L*.

b) Perform a similar verification of the equivalence of ⌈/*L* and ~⌊/~*L*.

c) Find similar relations among the functions <, ≤, =, ≥, and ≠. For example, ≠/*L* is equivalent to ~=/~*L*.

8.11 Evaluate the following:

```
V←1 2 3 4
M←(⍳3)∘.×⍳3

,M

+/,M

+/M

+/+/M

,V

V,V

V,,M
```

8.12 Evaluate the following:

```
A←2 3 5
B←1 3 5 7 9

ρA

ρB

+/A=A

+/B=B


M←A∘.+B

ρM

×/ρM

ρ⍉M

ρB∘.+A


ρ,M

ρ,A

ρ2

ρ,2
```

# 9

9.1 Define a function called *D*6 to determine divisibility of its argument by 6. Then evaluate the following:

```
D6 12

D6 ⍳12

D6 (⍳10)∘.+(⍳10)

D6 (⍳10)∘.×(⍳10)

D6 (⍳10)∘.-(⍳10)
```

9.2 Define a function called *B* which determines the square of its argument. Then evaluate the following:

```
B ⍳6

B (⍳6)∘.+(⍳6)
```

9.3 Define a function called *R*7 which yields the remainder when its argument is divided by 7. Then evaluate the expression *R*7 ⍳12.

9.4 Define a function called *IQ*7 which yields the integer part of the quotient of its argument when divided by 7. Then evaluate the expression *IQ*7 3 74 23 49.

9.5 Using the functions defined in the preceding exercises, evaluate the following:

```
3×D6 ⍳10

+/D6 ⍳10

⌊/D6 72 138 252

3×B 2+⍳5


X←12+2×⍳8

7×IQ7 X

(7×IQ7 X)+R7 X
```

9.6 a) Using the functions defined in preceding exercises, evaluate the expression *D*6 *R*7 *B* ⍳8.

b) Let *C* be the function defined as follows:

```
∇Z←C X
Z←D6 R7 B X∇
```

Now evaluate the expression *C* ⍳8.

9.7 Define monadic functions to yield each of the following results:

a) The area of a square as a function of the length of its side.

b) The area of a circle as a function of its radius (Use 3.1416 as an approximation to pi).

c) The area of a circle as a function of its diameter.

d) The volume of a sphere as a function of its radius.

e) The length of a rope in inches as a function of its length in feet.

9.8 Use the dyadic function *F* defined in the text to evaluate the following:

```
2 4 6 8 F 13 14 15 16

4 F 13 14 15 16

2 4 6 8 F 13

M←(⍳5)∘.+(⍳5)

M F 7+M
```

9.9 Define a dyadic function called *H* which gives the area of the rectangle whose length is given by the first argument and whose width is given by the second argument. Then evaluate the following:

```
3 H 4

3 4 5 H 5 6 7

3 H 5 6 7

3 4 5 H 5
```

9.10 Define a dyadic function *K* which yields the volume of the square cylinder, where the first argument represents the height of the cylinder and the second argument represents the length of the square base.

9.11 Define dyadic functions to yield each of the following results (the first argument mentioned is to be the first argument of the function):

a) The area of a triangle as a function of its base and altitude.

b) The perimeter of a rectangle as a function of its length and width.

c) The width of a rectangle as a function of its area and length.

d) The width of a rectangle as a function of its length and area.

e) The volume of a circular cylinder as a function of its height and the radius of its base.

f) The altitude of a triangle as a function of its area and base.

9.12 a) A rectangular plot is to be enclosed with 432 yards of fencing. Define a function to give the area of the enclosed plot (in square yards) as a function of the length of one of the sides (in yards).

b) Evaluate the function for a number of arguments to determine that value which yields the largest possible area.

9.13 a) A rectangular plot is to be enclosed with a fence of length *L*. Define a function which gives the area enclosed as a function of *L* and of the length *S* of one of the sides.

b) Evaluate the function for a number of values of *L* and *S* and determine the largest possible value of the area for a given fence length *L*.

c) How do the values of *L* and *S* compare when *S* has been chosen to give maximum area for some fixed value of *L*?

9.14 Using the function *PR* defined in the text, determine the value of the expression ⍴*PR X* for the following values of *X*: 10, 15, and 20.

9.15 Using the functions *FTOC* and *CTOF* defined in the text, evaluate the following:

```
FTOC 20+⍳10

CTOF FTOC 20+⍳10

CTOF 20+⍳10

FTOC CTOF 20+⍳10
```

9.16 Using the function *A* defined for adding rationals, evaluate the following:

```
3 4 A 1 2

÷/3 4 A 1 2

(÷/3 4)+(÷/1 2)

5 7 A 4 6

21 3 A 15 8

27 7 A 1 10

14 13 A 26 29
```

9.17 Define a function *M* which multiplies rationals in the same manner that the function *P* adds them. Then evaluate the following:

```
3 4 M 1 2

÷/3 4 M 1 2

(÷/3 4)×(÷/1 2)

5 7 M 4 6

21 3 M 15 8

27 7 M 1 10
```

9.18 Define a function *D* which divides one rational by a second. Then evaluate the following:

```
3 4 D 2 1

÷/3 4 D 2 1

(÷/3 4)÷(÷/1 2)

5 7 D 4 6
```

9.19 Using the function *R* of the text, show the results produced by the following execution traces:

```
TΔR←⍳4
Q←R 3
Q←R 4

TΔR←2 4
Q←R 3
Q←R 4
```

# 10

10.1 Analyze each of the four following function tables, that is, determine a function to fit each table:

| | |
|---|---|
| 0 | .4 |
| 1 | 2.1 |
| 2 | 3.8 |
| 3 | 5.5 |
| 4 | 7.2 |
| 5 | 8.9 |

| | |
|---|---|
| 0 | ¯3.9 |
| 1 | ¯2.7 |
| 2 | ¯1.5 |
| 3 | ¯0.3 |
| 4 | 0.9 |
| 5 | 2.1 |

| | |
|---|---|
| 0 | ¯4.7 |
| 1 | ¯1.9 |
| 2 | 0.9 |
| 3 | 3.7 |
| 4 | 6.5 |
| 5 | 9.3 |

| | |
|---|---|
| 0 | 15 |
| 1 | 19 |
| 2 | 23 |
| 3 | 27 |
| 4 | 31 |
| 5 | 35 |

**10.2** For each of the tables of Exercise 10.1 make a corresponding map and use it to determine an expression representing the table. Compare the results with the results of Exercise 10.1.

**10.3** Graph each of the functions of Exercise 10.1.

**10.4** Graph each of the following two functions:

| | |
|---|---|
| 0 | ¯12.4 |
| 1 | ¯8.9 |
| 2 | ¯5.6 |
| 3 | ¯2.5 |
| 4 | 0.4 |
| 5 | 3.1 |
| 6 | 5.6 |
| 7 | 7.9 |
| 8 | 10.0 |
| 9 | 11.9 |
| 10 | 13.6 |
| 11 | 15.1 |
| 12 | 16.4 |
| 13 | 17.5 |
| 14 | 18.4 |
| 15 | 19.1 |
| 16 | 19.6 |
| 17 | 19.9 |
| 18 | 20.0 |
| 19 | 19.9 |
| 20 | 19.6 |
| 21 | 19.1 |
| 22 | 18.4 |
| 23 | 17.5 |
| 24 | 16.4 |
| 25 | 15.1 |
| 26 | 13.6 |
| 27 | 11.9 |
| 28 | 10.0 |
| 29 | 7.9 |
| 30 | 5.6 |
| 31 | 3.1 |
| 32 | 0.4 |
| 33 | ¯2.5 |
| 34 | ¯5.6 |
| 35 | ¯8.9 |
| 36 | ¯12.4 |
| 37 | ¯16.1 |
| 38 | ¯20.0 |
| 39 | ¯24.1 |

| | |
|---|---|
| 0 | ¯61 |
| 1 | ¯50.59 |
| 2 | ¯41.32 |
| 3 | ¯33.13 |
| 4 | ¯25.96 |
| 5 | ¯19.75 |
| 6 | ¯14.44 |
| 7 | ¯9.97 |
| 8 | ¯6.28 |
| 9 | ¯3.31 |
| 10 | ¯1.00 |
| 11 | 0.71 |
| 12 | 1.88 |
| 13 | 2.57 |
| 14 | 2.84 |
| 15 | 2.75 |
| 16 | 2.36 |
| 17 | 1.73 |
| 18 | 0.92 |
| 19 | ¯0.01 |
| 20 | ¯1.00 |
| 21 | ¯1.99 |
| 22 | ¯2.92 |
| 23 | ¯3.73 |
| 24 | ¯4.36 |
| 25 | ¯4.75 |
| 26 | ¯4.84 |
| 27 | ¯4.57 |
| 28 | ¯3.88 |
| 29 | ¯2.71 |
| 30 | ¯1.00 |
| 31 | 1.31 |
| 32 | 4.28 |
| 33 | 7.97 |
| 34 | 12.44 |
| 35 | 17.75 |
| 36 | 23.96 |
| 37 | 31.13 |
| 38 | 39.32 |
| 39 | 48.59 |

**10.5** Use the graphs of Exercise 10.3 to analyze each of the functions they represent. Compare the results with those of Exercise 10.1.

**10.6** Consider the function *L* as defined below:

```
∇Z←C L X
Z←C[1]+C[2]×X∇
```

When applied to any two-element vector left argument and any vector right argument it produces a function which plots as a straight line. For example, if *X*←0,⍳5, then *X* is the first column of the first table of Exercise 10.1 and .4 1.7 *L X* is the second column.

a) Write expressions using *L* to produce the second column of each of the tables of Exercise 10.1.

b) Use the function *L* to produce a number of new function tables. Then graph each function and use the graph to analyze the function (i.e., determine an expression for it). It is best if you do not know or remember the expression which produced the table - either exchange tables with fellow students or lay your tables aside for a few days before analyzing them.

**10.7** Use the graphs produced in Exercise 10.4 to answer the following questions about each of the functions they represent:

a) For what value (or values) of the argument does the function have the value 0?

b) For what values of the argument is the function equal to 3, to ¯3, to 100?

c) For what argument values does the function reach a local high point?

d) For what argument value does the function appear to be changing most rapidly.

10.8 For each of the function tables of Exercise 10.4 attempt to find an expression which represents the function. For each expression you try, evaluate it for some or all of the argument values in the table to see how closely your proposed function fits the given function. You may find some of the results of Exercise 10.7 useful.

10.9 Evaluate the following:

```
3↑⍳5

¯3↑⍳5

3↓⍳5

¯3↓⍳5

7↑⍳5

¯7↑⍳5


A←1 2 3 4 5
B←6 7 8

⍴A

⍴B

(⍴B)↑A

B+(⍴B)↑A

A+(⍴A)↑B
```

10.10 a) Evaluate the following:

```
Y←0 1 4 9 16 25 36

1↓Y

¯1↓Y

V←(1↓Y)-(¯1↓Y)

V


W←(1↓V)-(¯1↓V)

W

(1↓W)-(¯1↓W)
```

b) Repeat Part (a) with

```
Y←(0,⍳6)*3
```

c) Repeat Part (a) with *Y* specified as the column of Fahrenheit values from Table 10.1.

d) Repeat Part (a) with *Y* specified as the second column of the first table of Exercise 10.4.

e) Repeat Part (a) with Y←⍳8.

10.11 Make a difference table for each of the functions of Exercise 10.1.

10.12 Make a difference table for each of the function tables produced in Exercise 10.6.

10.13 Use the difference tables produced in Exercise 10.11 to determine expressions to fit each of the functions. Compare the results with those of Exercise 10.1.

10.14 Use the difference tables produced in Exercise 10.12 to determine expressions to fit each of the functions. Compare the results with those of Exercise 10.6.

10.15 Make a difference table for each of the functions of Exercise 10.4. Be sure to include enough columns in the table so that the last column has a constant value.

10.16 Use the difference tables of Exercise 10.15 to determine an expression for each of the functions represented. Evaluate your expressions for a few arguments (say, 0 5 10 20 30) to see if your expressions do properly represent the functions.

10.17 Extend each of the difference tables produced in 10.15 by appending two further columns. What can you say about any column which follows a constant column?

10.18 Consider the following function:

```
∇Z←C QUADRATIC X
Z←(X-C[1])×(X-C[2])∇
```

When applied to any two-element vector left argument and any vector right argument it produces a function called a <u>quadratic</u> function. Choose various values of the left argument and the value 0,ι6 for the right argument to produce tables for a number of functions. Make difference tables to analyze each of the functions produced and apply each of the expressions produced to the argument 0,ι6 to see if the expressions properly represent the functions.

10.19 Repeat Exercise 10.18, replacing the quadratic function by the <u>cubic</u> function defined as follows:

```
∇Z←C CUBIC X
Z←(X-C[1])×(X-C[2])×(X-C[3])∇
```

The left argument must, of couse, be a 3-element vector.

10.20 Extend one of the difference tables of Exercise 10.15 by one column (of zeros) to make two tables of the same size to be used as follows:

a) Multiply the first table by 3 and verify that the resulting table is a proper difference table.

b) Multiply the second table by 4 and verify that the result is a proper difference table.

c) Add the two tables and verify that the result is a proper difference table.

d) Add 3 times the first table to 4 times the second table and verify that the result is a proper difference table.

10.21 a) Use the difference table produced in Exercise 10.20(a) to determine an expression for the function it represents. Compare this expression with 3 times the expression produced in Exercise 10.16.

b) Repeat Part (a) for each of the difference tables produced in Exercise 10.20, comparing each result with an appropriate expression from the results of Exercise 10.16.

10.22 Evaluate the factorial polynomial of order 7 for the arguments 0,ι7 and from the results form the difference table for the polynomial.

10.23 Evaluate the following:

```
∇Z←G X
Z←¯3+X*2∇

X←¯4+ι7
X

V←G X
V

L←⌈/V
S←⌊/V
R←⌽(¯1+S)+ι1+L-S
R
```

```
M←R∘.=V
M
```

10.24 A logical table containing many zeros can be displayed more easily using squared paper, drawing lines to enclose a rectangle of the same shape as the table and entering a 1 in each square corresponding to a 1 element in the table. The zeros need not be entered. Display the matrix *M* of Exercise 10.23 in this manner.

10.25 a) Evaluate the following, using the scheme of Exercise 10.24 to display any logical tables produced:

```
∇Z←H X
Z←X*3∇

X←¯4+⍳7
V←H X
R←⌽(¯1+⌊/V)+⍳1+(⌈/V)-⌊/V
M←R∘.=V
M
```

b) Repeat Part (a), replacing each use of the function *H* by use of the following function *K*:

```
∇Z←K X
Z←(X-1)×(X+2)∇
```

10.26 Evaluate the following, using the scheme of Exercise 10.24 to display the logical tables produced:

```
X←¯9+⍳17

2>|X∘.-X

5<|X∘.-X

(2>|X∘.-X)⌈(5<|X∘.-X)

7≥|X∘.-X

7<|X∘.-X

6=X∘.+X

12=X∘.×X

12=|X∘.×X
```

10.27 Evaluate the following, using the scheme of Exercise 10.24 to display any logical tables produced:

```
X←0,.1×⍳10
V←X*2
R←0,.05×⍳20
W←|R∘.-V

.01≥W

.02≥W

.1≥W
```

10.28 Evaluate the following:

```
ALPH←'ABCDEFGHIJKLMNOPQRSTUVWXYZ'

ALPH[8 9 7 8]

ALPH[⍳4]

ALPH[⌽⍳4]

⌽ALPH[⍳4]

ALPH[6⍴24]
```

10.29 Evaluate the following, assuming that *ALPH* has the value assigned in Exercise 10.28:

```
B←'*⎕ +-×'

B[1+2|⍳9]

B[3-2|⍳7]

B[1+6|(⍳7)∘.+⍳7]

A←ALPH,' '
A[9 27 19 9 14 7 27 15 6]
```

10.30 Use the graphing function *GR* of Section 10.12 to evaluate the following:

```
X←⍳8
T←X∘.≤X

GR T

GR ⍉T

M←X∘.⌈X

GR 4<M

GR 5<M

GR ⊖⌽5<M

GR (5<M) ⌈ ⊖⌽5<M
```

10.31 Evaluate the following:

```
M←(⍳8)∘.⌈⍳8
C←' ∘-+×○⋆⎕'

C[M]

C[5⌊M]

C[5⌈M]

C[M⌈⊖⌽M]
```

# 11

11.1 The phrase "define *F* by the expression 3+4×*X*" will be used to mean "Define the function *F* as follows":

```
∇Z←F X
Z←3+4×X∇
```

a) Define *P* by the expression 8+4×*X*.

b) Define *Q* as the function inverse to *P*.

c) Evaluate the following:

```
Q 0,⍳5

P Q 0,⍳5

P 0,⍳5

Q P 0,⍳5
```

11.2 a) Define *F*1, *F*2, etc., by the following:

¯3+2×*X*

¯8+10×*X*

¯2+¯10×*X*

4+3×*X*

4×*X*

5+*X*

b) Define functions *G*1, *G*2, etc., which are inverse to the functions *F*1, *F*2, etc.

c) Evaluate the following:

*X*←¯3+⍳5

*F*1 *X*

*G*1 *F*1 *X*

*G*1 *X*

*F*1 *G*1 *X*

d) Repeat Part (c) for each of the other function pairs *F*2 and *G*2, *F*3 and *G*3, etc.

11.3 Take the four function tables of Exercise 10.1 and replace the first column of each by the vector 2 2.2 2.4 2.6 2.8 3. Analyze each of the functions represented by the new tables. Verify your work by applying each of the resulting expressions to the arguments 2 2.2 2.4 2.6 2.8 3.

11.4 Repeat Exercise 11.3 but replacing the first columns by each of the following vectors:

¯7 ¯4 ¯1 2 5 8

¯2.5 ¯1 0.5 2 3.5 5

11.5 Make maps to show the application of each of the pairs of inverse functions of Exercise 11.2.

11.6 Draw graphs to represent each of the pairs of inverse functions of Exercise 11.2.

11.7 Define *Q* by the expression *X*⋆3. Graph the function *Q* for argument values from ¯2.5 to 2.5. Draw the graph of the function *R* which is inverse to *Q* and use it to evaluate (approximately) the expression *R* ¯1.3 0 1.27 2.15. Check these results by applying the function *Q* to them.

11.8 Graph the function -*X* and from it obtain the graph for the inverse function. What is the expression for the inverse function?

11.9 Repeat Exercise 11.8 for the function ÷*X*.

11.10 The function *X*⋆2 is called the square function and its inverse is called the square root. Determine the square root of each of the arguments 3, 5, 6, and 4096. Check your results by applying the square function.

11.11 The expression *X*⋆3 is called the cube function and its inverse is called the cube root. Determine the cube root of each of the arguments 3, 5, 6, and 4096. Check your results by applying the cube function.

11.12 Solve the following:

```
5=3+X

7=4×X

18=4+3×X

248=13+2×X-3

164=¯8+(2×X)-8

164=¯8+(2×X)÷8
```

11.13 Solve the following:

```
5=X*2

6=X*3

4096=X*3

256=(X-4)*2

343=(X+15)*3
```

# 12

12.1 Show the complete trace of the first four iterations of the function *SQRT* (defined in the text) when applied to each of the arguments 5 and 25 and .25. Check the results by applying the square function to them.

12.2 Show the complete trace of the function *SQT* when applied to the arguments 5 and 25 and .25 (carry all calculations to 7 decimal digits.)

12.3 Show the complete trace of the execution of the expression 4 5 *GRF* 20 for the case where *F* is the square function.

12.4 Show the complete trace of the execution of the expression 3 2 *GRF* 3, where the function *F* is defined as follows:

```
∇ Z←F X
Z←5×(X-1.4)×(X-2.6)×(X-4.2)∇
```

12.5 Write an expression using the function *GRF* which would yield a solution to the equation

```
17=X*4
```

and show the appropriate definition of the function *F* used within *GRF*.

12.6 Repeat exercise 12.5 for each of the following:

```
29=(X-2)*3

265=X*5

19=(3+2×X)*2

47=(¯2+.5×X)*6
```

12.7 Show the complete trace of the execution of the following:

```
GCD 35 133

GCD 133 35

GCD 140 35

GCD 1728 840
```

12.8 a) Evaluate the expression *V÷GCD V* for each of the following values of the argument *V*:

```
  6    8

 35  133

 54  318

175 2025

1024  128
```

b) For each of the cases of Part (a) verify that *V* and *V÷GCD V* both represent the same rational number, that is, (÷/*V*)=(÷/*V*÷*GCD V*).

c) Apply the function *GCD* to each of the results of Part (a) to verify that the elements of the result have no common factor, that is, their greatest common divisor is 1.

12.9 a) Use the function *A* defined in Section 9.5 (to add rationals) to evaluate the following:

```
3 4 A 1 2

7 20 A 8 45

3 8 A 5 16

74 100 A 13 50
```

b) Apply the function *GCD* to each of the results of Part (a).

12.10 a) Define a dyadic function *PLUS* which adds two rationals (in the manner of the function *A* of Section 9.5), but which yields the result in "reduced form", that is, with the smallest integers possible. Use the functions *A* and *GCD* in the definition.

b) Redefine the function of Part (a) so that the functions *A* and *GCD* are <u>not</u> used within it but are each replaced by statements like those in their definitions.

12.11 Define a function *TIMES* which multiplies rationals and produces the result in reduced form.

12.12 Evalute the expression +/*BIN N* for integer values of *N* from 0 to 7. Give a simple expression which is equivalent to the function +/*BIN N* and test it by evaluating both expressions for the case *N*←12.

12.13 Evaluate the expression -/*BIN N* for values of *N* from 0 to 7. Give a simple expression which is equivalent to the function -/*BIN N*.

12.14 Each of the following functions is equivalent to some primitive function (although possibly only for non-negative integer arguments). Evaluate each for a few scalar arguments and identify the equivalent primitive function:

```
    ∇ Z←X A Y          ∇ Z←B X
[1]   Z←1          [1]   Z←1
[2]   →3×Y≠0       [2]   I←0
[3]   Y←Y-1        [3]   →4×I≠X
[4]   Z←X×Z        [4]   I←I+1
[5]   →2 ∇         [5]   Z←I×Z
                   [6]   →3 ∇

    ∇ Z←X C Y
[1]   Z←X
[2]   →3×X<Y
[3]   Z←Y ∇
```

12.15 Without using the complement function (~) itself, define a function *D* which is equivalent to the complement function.

12.16 Repeat Exercise 12.15 for each of the following functions:

Minimum (⌊)

Magnitude (|)

Not-equal (≠)

12.17 a) Without using the residue function (|) itself define a function equivalent to the residue function, at least for non-negative right and left arguments.

b) Modify the function defined in Part (a) so that it is equivalent to the residue function for negative as well as positive right arguments.

12.18 a) Use the ceiling function (⌈) to define a function equivalent to the floor function (⌊).

b) Without using any of the ceiling, floor, or residue functions, define a function which is equivalent to the floor function for non-negative arguments.

c) Modify the function defined in Part (a) to make it apply to negative as well as non-negative arguments.

12.19 Consider the function *W* defined as follows:

```
    ∇ Z←W N
[1]   Z←2
[2]   I←2
[3]   I←I+1
[4]   →5×I≤N
[5]   →6-3×∨/0=Z|I
[6]   Z←Z,I
[7]   →3 ∇
```

Evaluate *W N* for a few different values of *N* and state in words what the function *W* does. (For integer arguments greater than 1 it is equivalent to a function defined in an earlier chapter).

# 13

13.1 Evaluate the following:

```
A←1 2 3 4 5
B←5 4 3 2 1
```

```
+/A×B
```

```
+/A⌈B
```

```
⌊/A⌈B
```

```
+/A≤B
```

```
⌊/A≤B
```

```
⌈/A≤B
```

```
×/A-B
```

```
+/A|B
```

```
+/A*B
```

```
+/B*A
```

```
C←¯10 3 14 ¯8 0 2
D←5 ¯7 2 ¯6 ¯1 3
```

```
+/C×D

⌈/C⌊D

⌊/C⌈D

⌈/(|C)⌊(|D)

⌊/(|C)⌊(|D)

+/C≤D

+/C=D

+/C-D
```

13.2 State in words what the following expressions mean. For example, the first one means the number of positions in which the elements of *Q* exceed the corresponding elements of *P*:

```
+/P<Q

+/P=Q

⌊/P≠Q

⌈/P=Q

×/P+Q

⌈/P+Q
```

13.3 Rewrite each of the expressions of Exercise 13.1 in inner product form.

13.4 Evaluate the following:

```
P←2 3 5 7 11
E←2 0 2 0 1
F←1 1 1 1 0

P×.*E

P×.*F

P×.*E⌊F

P×.*E⌈F

2 3 5 7 11×.*2 0 2 0 1

1 0 1 1 0+.×⍳5

¯1 1 ¯1 ¯1 1+.×⍳5

(¯1*1 0 1 1 0)+.×⍳5


(¯1*1 0 1 1 0)+.×P

(P<7)+.×P

(P≠5)+.×P

(¯1*P≠5)+.×P


P⌊.=E

P⌊.=P

P⌊.=F

1 2 3 4⌊.=⍳4
```

13.5 Evaluate each of the following:

```
1 3 3 1+.×(5*0 1 2 3)

X←5
C←1 3 3 1
E←0 1 2 3

C+.×(X*E)

(X*¯1+⍳⍴C)+.×C

(X+1)*3


D←1 2 1

(X*¯1+⍳⍴D)+.×D

(X+1)*2


B←1 4 6 4 1

(X*¯1+⍳⍴B)+.×B

(X+1)*4
```

```
X←7

(X*¯1+⍳⍴D)+.×D

(X+1)*2

(X*¯1+⍳⍴C)+.×C

(X+1)*3

(X*¯1+⍳⍴B)+.×B
```

13.6 Evaluate the following:

```
X←1 2 3 4 5 6
A←3⍴X

A

⍴A


B←8⍴X

B

⍴B

1,(5⍴4 2),1

7⍴1 0
```

13.7 Evaluate the following:

```
X←1 2 3 4 5 6
M←2 3⍴X

M

⍴M


N←3 5⍴X

N

⍴N

4 3⍴⍳12

⍉3 4⍴⍳12

⍴4 3⍴⍳12

⍴⍉4 3⍴⍳12
```

13.8 Let *M* and *N* be the following matrices:

```
    M                  N

 4 ¯6  3          2  0  5 ¯1
¯1  0 ¯4         ¯3  1  6  0
                  1 ¯2  3 ¯4
```

Then evaluate the following:

```
M+.≤N

M⌊.+N

M+.⌊N

M+.×N

⍉(⍉N)+.×(⍉M)


M+.=N

⍉(⍉N)+.=⍉M

M+.≤N

⍉(⍉N)+.≤(⍉M)

⍉(⍉N)+.>(⍉M)
```

13.9 State in words what each of the first six expressions of the preceding exercise represent.

13.10 Let *Q* and *C* be specified as follows:

```
Q←1 5⍴0 1 2 3 4 and
C←5 5⍴1 1 1 1 1 0 1 2 3 4 0 0 1 3
      6 0 0 0 1 4 0 0 0 0 1
```

Then *Q* and *C* are the following matrices:

```
     Q                C
0 1 2 3 4        1 1 1 1 1
                 0 1 2 3 4
                 0 0 1 3 6
                 0 0 0 1 4
                 0 0 0 0 1
```

Now evaluate the following:

```
X←3

(X*Q)+.×C

(X+1)*Q


X←4

(X*Q)+.×C

(X+1)*Q

(7*Q)+.×C

(7+1)*Q
```

13.11 Evaluate the following:

```
M←(⍳5)∘.≤⍳5

M


X←2 3 5 7 11

X+.×M

(+/1↑X),(+/2↑X),(+/3↑X),
     (+/4↑X),(+/5↑X)

M+.×X

(⍉M)+.×X


X×.*M

(×/1↑X),(×/2↑X),(×/3↑X),
    (×/4↑X),(×/5↑X)

X×.*⍉M
```

13.12 Let the matrices *I* and *D* be defined as follows:

```
    I                   D

1 0 0 0 0          1  0  0  0  0
0 1 0 0 0         ¯1  1  0  0  0
0 0 1 0 0          0 ¯1  1  0  0
0 0 0 1 0          0  0 ¯1  1  0
0 0 0 0 1          0  0  0 ¯1  1
```

Then evaluate the following:

```
X←2*⍳5

X

I+.×X

I+.×14 3 16 ¯7 0

D+.×X

D+.×14 3 16 ¯7 0
```

13.13 a) Write an expression using outer product to define the matrix *I* of Exercise 13.12.

b) Write an expression using outer products to define the matrix *D* of Exercise 13.12.

c) Modify the expressions derived in Parts (a) and (b) to define similar matrices of any specified dimension *N*.

d) The expression *I*+.×*X* is a function of the vector *X*. State in words what this function is.

e) The function *D*+.×*X* is closely related to the difference function defined in Section 10.6. State exactly what this relationship is.

f) State in words how the matrix *D* should be modified to produce a matrix *D*1 such that the function *D*1+.×*X* is exactly the difference function of Section 10.6.

g) Write an expression using outer products to define the matrix *D*1 of part (f).

13.14 Let $D$ be the matrix defined in Exercise 13.12, and let $S$ be the following matrix:

```
    S
1 0 0 0 0
1 1 0 0 0
1 1 1 0 0
1 1 1 1 0
1 1 1 1 1
```

a) Evaluate the following:

```
X←1 4 9 16 25

D+.×X

S+.×(D+.×X)

S+.×D

(S+.×D)+.×X


S+.×X

D+.×(S+.×X)

D+.×S

(D+.×S)+.×X
```

b) State in words the relation between the functions `D+.×X` and `S+.×X`.

13.15 Let $M$ be the following matrix:

```
   M
2 3 1 4
0 1 2 0
2 3 2 4
0 1 0 0
1 0 0 1
```

a) Evaluate the following:

```
P←2 3 5 7 11
N←P×.*M

N

GCD←P×.*⌊/M

GCD

N÷GCD
```

b) Verify that `GCD` is the greatest common divisor of the elements of `N`.

c) Choose any other value for `M`, except that the matrix must have 5 rows and must contain only non-negative integer elements. Then repeat Parts (a) and (b).

13.16 a) Using the matrix `M` of Exercise 13.15, evaluate the following:

```
P←2 3 5 7 11
N←P×.*M

N

LCM←P×.*⌈/M

LCM

LCM÷N
```

b) Verify that `LCM` is the least common multiple of the elements of `N`.

c) Choose another value for `M` (as in Exercise 13.15 (c)) and repeat Parts (a) and (b).

13.17 Let $M$ be the matrix:

```
     M
2    3   ¯5
0    1    2
4   ¯2    2
```

a) Evaluate the following:

```
A←(M[;1]×2)+(M[;2]×1)+(M[;3]×3)
B←M+.×2 1 3
V←2 ¯4 3
C←(M[;1]×V[1])+(M[;2]×V[2])
      +(M[;3]×V[3])
D←M+.×V
```

b) Display and compare the values of *A* and *B* and of *C* and *D*. State in words the relationship this comparison suggests.

c) Test the relationship you expressed in Part (b) by evaluating *C* and *D* for several different values of *V* and of *M*.

13.18 Follow the steps of Exercise 13.17 to establish a similar relationship between the expression *V*+.×*M* and expressions involving the rows of *M*.

13.19 a) Evaluate the following:

```
X←4

X*0 1 2 3

5 2 0 1×X*0 1 2 3

+/5 2 0 1×X*0 1 2 3


E←0 1 2 3

+/5 0 0 0×X*E

+/0 5 0 0×X*E

5×X*1

+/0 0 7 0×X*E

7×X*2
```

b) Identify each of the curves of Figure 13.2, labelling each as a "first term", "second term", etc.

13.20 Let the functions *SUM* and *TERMS* be defined as follows:

```
  ∇ Z←SUM X
[1]  Z←+/X∇
```

```
  ∇ Z←C TERMS X
[1]  Z←C×X*¯1+ιρC∇
```

Evaluate the following:

```
C←2 1 0 4
X←5

C TERMS X

SUM C TERMS X
```

13.21 Repeat Exercise 13.20 for the following values of *X* and *C*:

| X | C |
|---|---|
| 4 | 1 3 3 1 |
| 5 | 0 0 0 1 |
| 5 | 1 3 3 1 |
| 6 | 0 0 0 1 |
| 0 | 1 3 3 1 |
| 1 | 0 0 0 1 |
| 2 | 1 4 6 4 1 |
| 3 | 0 0 0 0 1 |

13.22 Use the function *P* defined in Section 13.6 to evaluate the following:

```
5 0 7 2 P 0 1 2 3 4 5

¯5 0 ¯7 2 P 0 1 2

¯5 0 ¯7 2 P ¯4 ¯3 ¯2

1 1 P 0,ι5

0 1 P 1+0,ι5

1 2 1 P 0,ι5

0 0 1 P 1+0,ι5

1 3 3 1 P 0,ι5

0 0 0 1 P 1+0,ι5
```

13.23 Use the difference function *D* defined in Section 10.6 to evaluate the following:

```
V←5 0 2 3 P 0,⍳5

V

D V

D D V

D D D V


W←5 0 2 3 1 P 0,⍳7

W

D D D D W
```

13.24 Use the matrix *S* defined in Exercise 13.14 to evaluate the following:

```
N←⍳5

S+.×N

0 0.5 0.5 P N

S+.×N*2

(0 1 3 2÷6) P N


S+.×N*3

0 0 0.25 0.5 0.25 P N

S+.×N*0

0 1 P N
```

# 14

14.1 For each of the dyadic functions + - × ÷ < ≤ = | ⌈ and ⌊, state:

a) Whether you think it is commutative or not.

b) An example proving that the function is non-commutative for each case you declare to be non-commutative.

14.2 Modify the function *COM* defined in Section 14.2 so as to include in its domain all of the function symbols appearing in Exercise 14.1.

14.3 a) Make tables to prove that the functions and and or are commutative.

b) Evaluate the following:

```
0∧1

0∨1

0⍲1

0⍱1


X←0 0 1 1
Y←0 1 0 1

X∧Y

X∨Y

X⍲Y

X⍱Y
```

14.4 Use the method of exhaustion to examine the commutativity (or non-commutativity) of the dyadic functions < ≤ = ≥ > ≠ and ⌊.

14.5 Make a table similar to Table 14.5 to prove that the minimum function is associative.

14.6 Make a table (of 8 cases labelled 0 0 0 and 0 0 1 and 0 1 0, etc., to 1 1 1) which will show whether the <u>and</u> function is associative.

14.7 Repeat Exercise 14.6 for each of the following functions: ∨ ⍲ ⍱.

14.8 a) Write an example to show that addition does not distribute over multiplication.

b) Write an example to show that addition does not distribute over itself.

c) Write an example to show that multiplication does not distribute over itself.

d) Write a few examples to <u>illustrate</u> that multiplication distributes over addition (include some negative numbers in the example).

e) Complete the following table so as to summarize the foregoing results, using a 1 to denote distributivity and a 0 to denote non-distributivity:

| | + | × |
|---|---|---|
| + | | |
| × | | |

14.9 Extend the table of Exercise 14.8 (e) to include the functions + × - ⌈ and ⌊. You are not expected to provide proofs of distributivity, but test the matter thoroughly by evaluating a number of expressions looking for values which will prove non-distributivity. Be sure to use some negative values in this search. For each case stated to be non-distributive, give an example which proves it so.

14.10 Make tables to determine whether:

a) ∨ distributes over ∧

b) ∨ distributes over ∨

c) ∧ distributes over ∧.

14.11 Summarize the results of Table 14.6 and of Exercise 14.10 in a distributivity table of the form

| | ∨ | ∧ |
|---|---|---|
| ∨ | | |
| ∧ | | |

entering a 1 in the *I*th row and *J*th column of the table if the function heading the *I*th row distributes over the function heading the *J*th column, and a 0 otherwise.

14.12 Extend the distributivity table of Exercise 14.11 to include the functions ∨ ∧ ⍲ and ⍱. Make tables of the form of Table 14.6 to develop any results you may need for this table.

14.13 a) Make a table similar to Table 14.7 to prove that addition distributes over maximum.

b) make a table to test whether subtraction distributes over maximum.

c) If in Exercise 14.9 you concluded that multiplication distributes over maximum, then evaluate the following pair of expressions and compare the results:

```
¯6×4⌈9

(¯6×4)⌈(¯6×9)
```

14.14 Repeat Exercise 14.13 substituting minimum for maximum.

14.15 Make a table of the form of Table 14.8 to summarize all of the results obtained thus far. Enter 0's and 1's <u>only</u> for results that have been <u>proven</u>, and leave other entries blank. Include the dyadic functions `+ - × ÷ ⌈ ⌊ ∨ ∧ ⍱` and `⍲`. Fill out blank spaces in the table by constructing further proofs if you wish.

14.16 Illuminate the proof given in Section 14.5 for each of the following values of *A*, *B*, and *C*:

| *A* | *B* | *C* |
|---|---|---|
| 3 | 14 | ¯8 |
| ¯3 | 5 | 7 |
| ¯3 | ¯5 | ¯7 |

14.17 a) Prove that `(P⌊Q)⌈R` is equivalent to `(R⌈P)⌊(Q⌈R)`. Use the first such proof in Section 14.5 as a model, writing the justification of each step to the right of it.

b) Choose values of *P*, *Q*, and *R* and illuminate the proof of Part (a).

14.18 Repeat Exercise 14.17 to show the equivalence of each of the following pairs of expressions:

```
A∧(B∧C)
C∧(B∧A)

A+(B+C)
C+(B+A)

A×B×C×D
D×C×B×A
```

14.19 For each of the proofs of Exercises 14.17 and 14.18 add the abbreviated form of the note to the right of each note in the proof.

14.20 Choose values of *A*, *B*, *C*, and *D* and use them to illuminate the proof (given in the text) that `(A+B)×(C+D)` is equivalent to `(A×C)+(A×D)+(B×C)+(B×D)`

14.21 Make (and illuminate) proofs for the following pairs of equivalent statements:

```
(A⌊B)+(C⌊D)
(A+C)⌊(A+D)⌊(B+C)⌊(B+D)

A∧(B∨C∨D)
(A∧B)∨(A∧C)∨(A∧D)
```

14.22 a) Determine a value of the vector *C* such that the expression `+/C×X*0 1 2` is equivalent to the expression `×/X+4 1`.

b) Evaluate the expressions in Part (a) for several values of *X* and compare the results (which should agree).

14.23 Repeat Exercise 14.22 for each of the following:

```
(X+4)×(X+1)

×/X-4 1

×/X+1 1

×/X+1 0

×/X+0 1

(X+¯1)×(X+¯1)
```

```
(X-1)×(X-1)


R←3 5

×/X+R

×/X+(-R)

×/X-R
```

14.24 Choose vector values of the arguments to illuminate the proof illuminated in Exercise 14.16.

14.25 Chose vector values to illuminate each of the proofs of Exercise 14.18.

14.26 Evaluate the following:

```
A←3 ¯8 15 6
B←¯5 0 18 43 ¯7

+/A,B

(+/A)+(+/B)

⌈/A,B

(⌈/A)⌈(⌈/B)

⌊/A,B


(⌊/A)⌊(⌊/B)

×/A,B

(×/A)×(×/B)

-/A,B

(-/A)-(-/B)


C←1 0 1 0 1
D←0 1 1

∨/C,D

(∨/C)∨(∨/D)
```

14.27 Evaluate the following:

```
A←3 ¯8 15 6
B←4 2 ¯1 4

+/A+B

(+/A)+(+/B)

×/A×B

(×/A)×(×/B)


⌈/A⌈B

(⌈/A)⌈(⌈/B)

-/A-B

(-/A)-(-/B)
```

14.28 Use each of the following pairs of values of *V* and *W* to illuminate the identity expressed by Theorem 4:

| *V* | *W* |
|---|---|
| 1 10 2 3 | 2 0 5 |
| 2 0 5 | 1 5 2 3 |
| ¯3 10 2 ¯8 | 2 0 ¯2 ¯3 1 |

14.29 Use the following values to illuminate Theorem 5:

```
A← 3 ¯1 0 4 2
B← 5 ¯2 6
P←¯2 2 1 0 5
Q← 7 2 ¯4
```

14.30 a) Repeat Exercise 14.29, substituting the function + for every occurrence of × in Theorem 5.

b) Repeat Part (a) using ⌈ instead of +.

14.31 Use the values of *A*, *B*, *P*, and *Q* from Exercise 14.29 and the values *I*←4 and *J*←2 to illuminate the proof of Theorem 5.

14.32 Use the following sets of values of *A*, *B*, and *C* to illuminate Theorem 6:

| *A* | *B* | *C* |
|---|---|---|
| 3 | 2 | 4 |
| ¯2 | 3 | 5 |
| 3 | 4 | ¯4 |

14.33 Choose some values for *X*, *E*, and *F* and use them to illuminate Theorem 7.

# 15

15.1 a) For each of the following pairs of values of *A* and *B*, determine a vector *D* such that the expression *D P X* is equivalent to (*A P X*)+(*B P X*) (where *P* is the polynomial function defined in Section 13.6):

| *A* | *B* |
|---|---|
| 2 1 4 | 3 ¯2 5 |
| 6 18 4 2 | ¯3 ¯3 8 ¯4 |
| 2 0 4 8 | 0 0 0 2 |

b) Verify each of the foregoing results by evaluating the expressions *D P X* and (*A P X*)+(*B P X*) for *X*←¯3+⍳5.

15.2 Repeat Exercise 15.1 for the following values of *A* and *B*:

| *A* | *B* |
|---|---|
| 6 1 2 | 3 0 ¯4 8 2 |
| 2 1 3 ¯2 4 | 2 0 1 |

15.3 a) Determine the coefficients of the polynomial equivalent to the product (*C P X*)×(*D P X*) for the following pairs of values of *C* and *D*:

| *C* | *D* |
|---|---|
| 3 ¯1 4 | 2 0 4 1 |
| 2 0 4 1 | 3 ¯1 4 |
| 3 ¯5 2 | 0 1 |
| 3 ¯5 2 | 0 0 1 |
| 3 ¯5 2 | 3 |

b) Verify the foregoing results for the case *X*←2.

15.4 Repeat Exercise 15.3 for the following pairs of values of *C* and *D*:

| *C* | *D* |
|---|---|
| 2 0 4 1 | 1 1 |
| 1 1 | 1 1 |
| 1 2 1 | 1 1 |
| 1 3 3 1 | 1 1 |
| 1 4 6 4 1 | 1 1 |

15.5 Illuminate the summary of the formal proof of Section 15.3 for each of the following sets of values of *C*, *D*, and *X*:

| *C* | *D* | *X* |
|---|---|---|
| 2 0 4 | 3 1 2 5 | 2 |
| 2 0 4 | 3 1 2 5 | 3 |
| 3 1 2 5 | 2 0 4 | 3 |

15.6 a) For each of the following determine the coefficients of an equivalent polynomial:

```
×/X+2 3

×/X+4 7

×/X+7 4

×/X+(-7 4)


×/X-7 4

×/X+¯7 ¯4

×/X+2 3 4

×/X+4 3 2


×/X+3 2 4

×/X-0 1

×/X-0 1 2

×/X-0 1 2 3
```

b) Verify each of the results of part (a) for the case *X*←5.

15.7 Perform a complete trace of the function *QA* of Section 15.4 for each of the following cases:

```
QA 2 1 3
QA 1 1 1 1
```

15.8 a) For each of the following determine the coefficients of an equivalent polynomial:

```
×/X+1

×/X+1 1

×/X+1 1 1

×/X+4⍴1

×/X+5⍴1

×/X+6⍴1
```

b) Compare the results of Part (a) with the binomial coefficients of Section 12.4.

15.9 Use the results of Exercise 15.7 to test the identity (×/*X*+*V*)=(*QA V*)*P X* for the following cases:

```
      V            X
  2   1   3        4
  1   1   1   1    3
```

15.10 The dyadic function ⌽ applied to a scalar left argument and a vector right argument rotates the vector to the left by an amount determined by the left argument. For example:

```
      3⌽1 2 3 4 5
4  5  1  2  3
```

If the left argument is negative, the rotation is to the right:

```
      ¯3⌽1 2 3 4 5
3  4  5  1  2
```

If the right argument is a matrix *M* and the left argument is a vector *V* whose size is equal to the number of rows of the matrix, then the *I*th row of *M* is rotated according to the value of *V*[*I*]. For example:

```
      M                     2 0 ¯2⌽M
  1  2  3  4  5         3  4  5  1  2
  9  8  7  6  5         9  8  7  6  5
  4  5  6  7  8         7  8  4  5  6
```

a) Use the foregoing definition of rotation to evaluate the expression

```
3 1 4 G 2 0 5 3
```

where *G* is defined as follows:

```
∇Z←C G D
Z←+/⍉(1-⍳⍴C)⌽C∘.×D,0×1↓C∇
```

b) Compare the result of part (a) with the polynomial product shown at the beginning of Section 15.3, and show that the function *G* produces the product of two polynomials.

15.11 Use the results of Exercise 15.8 to test the identity `((X+1)*N)=(Q N⍴1)P X` for the case `X←3` and the cases *N* equal to 1, 2, 3, and 4.

15.12 Perform complete traces of the functions *BIN* and *QA* (shown in Section 15.5) for the following cases:

```
QA 4⍴1
BIN 4
```

15.13 Let *M* be the following matrix:

```
1 0  0  0
0 1 ¯1  2
0 0  1 ¯3
0 0  0  1
```

a) Compare the <u>columns</u> of *M* with the coefficients of polynomials equivalent to the factorial polynomials and state how the columns correspond to the degrees of the factorial polynomials. (Note that final zeros appended to a vector of coefficients make no difference to the value of the polynomial).

b) Evaluate the following:

```
V←0,1,(3÷2),(2÷6)
A←M+.×V
A
```

c) Use the results of Exercise 13.17 (in Chapter 13) to state in words the relation between the result of Part (b) and a certain weighted sum of the columns of *M* (that is, of the coefficients of polynomials equivalent to the factorial polynomials).

d) Use the vector *A* of Part (b) and the polynomial function *P* defined in the text to evaluate the expression `A P X` for several values of *X*. Compare the results with the evaluation of `+/(⍳X)*2` for the same values of *X*.

e) Explain the agreements obtained in Part (d).

15.14 Exercise 15.13 illustrated how the expression `M+.×V` would yield the coefficients of a polynomial equivalent to the sum of `V[1]` times the 0-degree factorial polynomials, `V[2]` times the 1-degree factorial polynomial, etc. Apply this result to obtain the coefficients of a polynomial equivalent to `+/(⍳X)*3` as follows:

a) Extend the matrix *M* to be a 5 by 5 matrix incorporating the coefficients for the next factorial polynomial.

b) Evaluate `+/(⍳X)*3` for a number of values of *X* beginning with 0.

c) Use the difference table method of Section 10.8 to determine an equivalent function (expressed as a weighted sum of factorial polynomials).

d) Evaluate the expression `Q←M+.×R÷!0 1  2 3 4`, where *R* is the first row of the difference table.

e) Compare `Q  P X` and `+/(⍳X)*3` for a number of values of *X*.

15.15 Use mathematical induction to prove that the functions `+/(⍳X)*2` and `(+/0 1 3 2×X*0 1 2 3)÷6` are equivalent.

# 16

16.1 Examine each of the number systems of Table 16.1 and then:

a) Add to each the representation of the next integer in sequence (that is, 18).

b) Repeat part (a) for the integer 19. State clearly how a particular difficulty arising in one of the systems is resolved.

c) For each system give a verbal statement of how it works.

d) Can the number 0 be represented in the Prime Factor System?

16.2 The vectors

```
V←1 0 0 0 0 0 0
W←1 1 0 0 0 0 0
X←2 1 0 0 0 0 0
```

are a triple of vectors occurring in Table 16.1 such that *X* is equal to *V*+*W*. Determine every triple of vectors satisfying these requirements and for each triple show that the product of the numbers represented by *V* and *W* is the number represented by *X*.

16.3 Evaluate the following:

```
V←2 3 5 7 11
R←2 1 4 1 3 4 7
A←'ABCDEFGHIJKLMNOPQRSTUVWXYZ'
T←'NOW IS THE TIME'

V⍳5
V⍳13
V⍳0
V⍳⍳⍴V
(⍳⍴V)⍳V

R⍳⍳⍴R
A⍳T
A[A⍳T]
```

16.4 a) Apply the function *TO*4 to the following arguments:

```
'HB'
'JJ'
'BCDE'
```

b) Apply the function *TO*1 to each of the results of part (a).

16.5 a) Use Table 16.2 to determine the prime factor representations of the following numbers:

```
480
512
960
111
139
125
```

b) Use Table 16.3 to determine the decimal value of the numbers represented by the following vectors:

```
5 1 1 0 0  0 0 0 0 0
9 0 0 0 0  0 0 0 0 0
6 1 1 0 0  0 0 0 0 0
0 0 3 0 0  0 0 0 0 0
1 1 1 1 0  0 0 0 0 0
1 0 1 0 0  0 0 0 0 1
```

16.6 Use Tables 16.2 and 16.3 to determine the greatest common divisor of each of the following pairs of numbers:

```
480 and 660
375 and 960
726 and 455
735 and 539
```

16.7 a) Use Table 16.2 to determine the prime factor representation of the least common multiple of each of the following pairs of numbers:

```
240 and 336
960 and 64
480 and 660
735 and 539
465 and 341
```

b) Determine the decimal values of each of the results of part (a). Where possible use Table 16.3.

16.8 a) Use Table 16.2 to determine the prime factor representation of the product of each of the following pairs of numbers:

```
 24 and  15
 75 and   7
 85 and  11
252 and 252
240 and 275
```

b) Determine the decimal values of the results of part (a). Use Table 16.3 where possible.

16.9 a) Use Tables 16.2 and 16.3 to determine the following quotients:

```
429÷39
960÷24
578÷34
```

b) Define a dyadic function which when applied to vectors *V* and *W* will determine whether the number represented by *V* is divisible by the number represented by *W*.

16.10 Show a complete trace of the function *IVDVAL* of Section 16.3 for the following arguments:

```
 548
 176
   3
2147
```

16.11 a) Evaluate the expression *A+B* for each of the following pairs of vectors *A* and *B*:

```
      A          B
    2 3 5      1 4 4
    2 3 5      7 9 5
  3 0 4 8      4 9 4 1
  8 1 2 6      5 1 3 0
  2 1 7 4      3 8 2 6
```

b) Which of the results of part (a) are acceptable vector decimal representations?

c) Apply the function *VDVAL* of Section 16.3 to the argument *A+B* for each of the cases of part (a) and show that each result is indeed the sum of the corresponding numbers represented by *A* and *B*.

16.12 Show a complete trace of the function *VDADD* of Section 16.3 for each of the pairs of arguments of Exercise 16.11(a).

16.13 Repeat Exercise 16.12 for the function *SERIALDADD* of Section 16.3.

16.14 Apply the function *VBVAL* of Section 16.4 to the following arguments:

```
1 1 0
1 1 0 1
1 1 1 1
1 0 0 0 0
```

16.15 Show a complete trace of the function *IVBVAL* of Section 16.4 for the following arguments:

```
 6
13
31
32
```

16.16 a) Make a matrix of 8 rows and 3 columns which lists the 3-digit vector binary representations of the integers from 0 to 7. (Note that any binary vector can be extended by appending zeros to the left without changing the value it represents.)

b) Repeat part (a) for the numbers 0 to 15, making the matrix of the appropriate size.

c) Repeat part (b) for the numbers 0 to 31.

d) State how the matrix for the numbers 0 to 63 could be constructed from the matrix produced in part (c).

16.17 a) Define a function called *VBADD* which will add vector binary representations in the same sense that the function *VDADD* of Section 16.3 adds vector decimal representations.

b) Apply the function *VBADD* defined in part (a) to the pair of arguments shown on each line below:

```
1 0 1 0 1       0 0 1 1 1
0 0 0 0 1       0 1 1 1 1
```

16.18 a) Define a function called *SERIALBADD* (and modeled on the function *SERIALDADD* of Section 16.3) which adds vector binary representations.

b) Apply the function defined in part (a) to the argument pairs of Exercise 16.17(b).

16.19 a) Define functions called *VAL*, *IVAL*, *ADD*, and *SERIALADD* which are derived from the analogous functions of Section 16.3 by incorporating the variable *BASE* as suggested in Section 16.5.

b) Illustrate the use of each of the functions defined in part (a) by applying them to suitable arguments for the case *BASE*←3.

c) Repeat part (b) for the case *BASE*←7.

16.20 Define a function called *BT* (for <u>b</u>alanced <u>t</u>ernary) which will add numbers represented in the ternary system using weights ¯1, 0, and 1.

16.21 Determine sum and carry tables for addition in each of the following number systems:

a) Ternary (i.e., base 3)
b) Base 4
c) Base 5
d) Base 7
e) Balanced Ternary (Base 3 with weights ¯1, 0, and 1).

16.22 Repeat Exercise 16.21 for multiplication instead of addition.

16.23 a) Apply the function *NVBVAL* of Section 16.7 to the following arguments:

```
  0 1 0 1
  1 1 0 1
0 1 0 1 0
1 1 0 1 0
0 0 0 0 0
1 0 0 0 0
```

b) Define an analogous function for evaluating a vector binary representation in which the leading element determines the sign. Illustrate its use by applying it to the following arguments:

```
0 7 1 6
1 7 1 8
```

16.24 a) Apply the function *FVD* of Section 16.8 to the following arguments:

```
4 0 0 2 4
2 1 3 4
3 9 8 4
```

b) Write the ordinary decimal representation for each of the results of part (a).

16.25 Define a function called *IFVD* which is inverse to the function *FVD* of Section 16.8 and illustrate its use by applying it to the results of Exercise 16.24(a).

16.26 a) Determine the representation of each of the following rational numbers in the *RFVD* system described in Section 16.8:

```
17÷3
25÷11
1÷7
```

b) Rationals are often represented by placing bars over the last *P* digits, where *P* is the length of the repeating pattern. (For example, the number represented by 3 4 1 2 6 3 7 4 in the *RFVD* system would be shown as $12.6\overline{374}$). Show this representation for each of the cases of part (a).

16.27 a) Apply the function *RFVD* to each of the results of Exercise 16.26(a) so as to obtain a rational vector representation for each.

b) Verify the results of part (a) by carrying out the indicated divisions to obtain the same representation obtained in Exercise 16.26(b).

c) Verify the results of part (a) by using the *GCD* function on each result to reduce it to lowest terms. The resulting vectors should agree with the original arguments of division given in Exercise 16.26(a).

# 17

17.1 Examine each of the functions defined in Chapter 12 and identify each use of a proposition, including propositions which occur as portions of more complex expressions. Identify each proposition by writing it and citing Section number, function name, and line number.

17.2 a) Examine each of the expressions used in the text of Chapter 4 and list each expression (or portion thereof) which is a proposition, indicating the Section in which each occurs. Note that the functions $\lceil$ and $\lfloor$ produce propositions when applied to purely logical results, that is, 0's and 1's.

b) For each of the propositions in the list of part (a) which uses the functions $\lceil$ or $\lfloor$, write an equivalent logical expression which does not use them. (See Sections 4.9 and 14.2.)

17.3 Comment on the following statement: "A theorem is a proposition which is asserted to be true (i.e., have the value 1) for any possible value of its arguments. Consider, for example, the proposition $((0=2|X)\wedge 0=3|X)\leq 0=6|X$.

17.4 Evaluate the following:

$X\leftarrow 0\ 0\ 1\ 1$
$Y\leftarrow 0\ 1\ 0\ 1$

$X\leq Y$
$\sim X>Y$
$X>Y$
$\sim X\leq Y$
$X\neq Y$
$\sim X=Y$

17.5 Evaluate the following:

$X\leftarrow 0\ 0\ 1\ 1$
$Y\leftarrow 0\ 1\ 0\ 1$

$\sim X$
$\sim Y$
$\sim\sim X$

$X\wedge Y$
$(\sim X)\vee(\sim Y)$
$\sim(\sim X)\vee(\sim Y)$

$X\vee Y$
$(\sim X)\wedge(\sim Y)$
$\sim(\sim X)\wedge(\sim Y)$

$X\leq Y$
$\sim(\sim X)<(\sim Y)$
$X=Y$
$\sim(\sim X)\neq\sim Y$

17.6 For each of the first three of the six sets defined by English sentences at the beginning of Section 17.2, write a function definition (i.e., a proposition) which defines the set. Comment on any cases for which such definition proves to be impossible. Use the names $P1$, $P2$, and $P3$ for the functions defined.

17.7 Each of the propositions defined in Exercise 17.6 should apply to any numbers (perhaps yielding a zero for most) and therefore has an infinite domain. For practical purposes one may limit attention to a more restricted domain (of a finite number of elements) called the universe of discourse. This universe is often specified by a vector which lists its elements.

a) Apply each of the logical functions *P1* and *P2* defined in Exercise 17.6 to the following universe of discourse:

```
UD←¯10+⍳35
```

b) Evaluate the expressions (*P1 UD*)/*UD* and (*P2 UD*)/*UD*.

17.8 a) Define functions called *P4*, *P5*, and *P6* which define the last three of the six sets described at the beginning of Section 17.2.

b) Evaluate the following:

```
P4 1
P4 2
P5 'B'
P5 'U'
P6 'G'
```

17.9 a) Define functions *P4V*, *P5V*, *P6V* which are equivalent to the functions *P4*, *P5*, and *P6* of Exercise 17.8 except that they will apply properly to vector left arguments.

b) Evaluate the following:

```
UD2←.5×¯9+⍳17
T←'I SING OF OLAF'

P4V UD2
(P4V UD2)/UD2
P5V T
P6V T
(P5V T)/T
```

17.10 a) Evaluate the following:

```
A←1 2 3 4 5
B←9 7 5 3

A∊B
B∊A
~A∊B
~B∊A
```

b) Evaluate the following:

```
(A∊B)/A
(B∊A)/B
(~A∊B)/A
(~B∊A)/B
```

c) For each of the expressions of part (a), state in words the proposition which defines the set produced by the expression.

d) Choose one or more new values of *A* and *B* to illustrate that your answers to part (b) apply to any pair of sets *A* and *B*.

17.11 a) Evaluate the following:

```
S←2 3 4 5 6 7 8

S∘.×S
S∊S∘.×S
~S∊S∘.×S
(~S∊S∘.×S)/S
```

b) Evaluate the expression (~*S*∊*S*∘.×*S*)/*S* for the case *S*←1+⍳19. Skip the evaluations of all of the intermediate results if you can.

c) Identify the function defined in Chapter 9 which is equivalent to the following function:

```
      ∇Z←F X
[1]    S←1+⍳X
[2]    Z←(~S∊S∘.×S)/S∇
```

17.12 a) Show several different vectors which could be used to represent the set represented by the vector $V \leftarrow 2\ 3\ 5\ 7$.

b) Repeat part (a) for the vector $W \leftarrow$'*STEAM*'.

c) Indicate which of the following vectors represent the same set as does the vector $V$:

```
V[4 2 1 3]
V[2 1 3 1 4]
V[4 2 1 3 2 2 2]
V[4 2 2 3 4 4 2]
```

d) Indicate which of the following vectors represent the same set as does the vector $W$:

```
W[5 3 4 2 1]
W[2 3 4 5 5 4 2 3 1]
W[2 4 1 2 3 1]
```

e) state in words the conditions which an integer vector $I$ must satisfy in order that the vector $X[I]$ represent the same set as the vector $X$.

f) Define a dyadic function *SAMESET* which defines the proposition requested in part (e), that is, the expression $X$ *SAMESET* $I$ must yield 1 if $X[I]$ represents the same set as does $X$, and 0 otherwise.

17.13 a) Define a monadic function called *REDUCE* which yields the shortest possible vector which represents the same set as the argument, i.e., the result is the argument with duplications removed.

b) Evaluate the following:

```
REDUCE 2 1 4 7 1 2 2 8
REDUCE 3 3 3 3 3
```

17.14 Apply the following function to the arguments of the function *REDUCE* in Exercise 17.13 (b) and compare the results with those obtained in applying the function *REDUCE*:

```
    ∇Z←RD X
[1]  Z←((⍳⍴X)=X⍳X)/X∇
```

17.15 a) Use the intersection function $I$ defined in the text to evaluate the following:

```
A←1 2 3 4 5 6
B←10 8 6 4 2
C←3 4 6 12

A I B
B I A
A I C
C I A
A I(B I C)
(A I B)I C
```

b) Remembering that two different vectors may represent the same set, state whether you believe the intersection function to be associative? Is it commutative?

c) Choose a variety of values for $A$, $B$, and $C$ and evaluate the expressions of part (a) in order to further illustrate the conclusions you presented in part (b).

17.16 Repeat Exercise 17.15 replacing the intersection function $I$ by the less function $L$ (defined in the text) throughout.

17.17 Repeat Exercise 17.15 replacing the intersection function $I$ by the union function $U$ throughout.

17.18 a) Does the intersection function *I* distribute over the union function *U* in the sense that results represent the same sets? Evaluate appropriate expressions for various values of arguments in order to illuminate your conclusion.

b) Repeat part (a) for the question of whether *U* distributes over *I*.

17.19 Consider the following monadic function:

```
      ∇Z←BT N
[1]    Z←2|⌊(¯1+⍳2*N)∘.÷2*⌽¯1+⍳N∇
```

a) Evaluate the expression *BT* 3 and show that the result is equal to the matrix *TAB* given in the text.

b) Evaluate the expression *BT N* for several positive integer values of *N* to illustrate that the result can be considered either as:

i) the enumeration of the *N*-digit binary representations of the first 2⋆*N* non-negative integers, or as

ii) the logical vectors which will select all possible subsets from a vector of *N* elements.

c) Evaluate the following:

```
T←2 3 7 10

T+.×⍉BT⍴T
T×.*⍉BT⍴T
```

d) State in words the significance of each of the expressions evaluated in part (c).

17.20 a) Evaluate the following:

```
R←2 3 5

N←-R
T←BT ⍴N

C←((⌽0,⍳⍴N)∘.=+/T)+.×N×.*⍉T
```

b) Evaluate the following expressions (the last two will agree if your work is correct):

```
X←4

X-R
×/X-R
+/C×X*¯1+⍳⍴,C
```

c) The expression +/*C*×*X*⋆¯1+⍳⍴,*C* is the definition of a polynomial with coefficients *C* as defined in Section 13.6, and the expression ×/*X*-*R* is the definition of a polynomial in terms of its zeros or roots *R*. The work of part (a) is a computation of the coefficients *C* of a polynomial which has the roots prescribed by the vector *R*. The method used is based on Newton's symmetric functions. Redo parts (a) and (b) for various values of *R* and *X* and verify that the last two results of part (b) agree in every case.

# 18

18.1 For each of the following linear expressions, write an equivalent expression in terms of a single vector argument *V*, where `V←X,Y` or `V←X,Y,Z` or `V←W,X,Y,Z` as appropriate:

```
3+(4×X)+(5×Y)

¯4+(6×X)+7×Y

¯4+(6×Y)+7×X

3+(¯6×X)+0×Y

3+(¯6×X)


¯8+(0×X)+¯9×Y

¯8+¯9×Y

-(8+9×Y)

0+(3×X)+(¯6×Y)

(3×X)+(¯6×Y)

(3×X)-(6×Y)


4-(3×X)+7×Y

8+(2×X)+(5×Y)+(10×Z)

8+(2×X)+(0×Y)+(10×Z)

¯4+(2×X)+(10×Z)

18+10×Z

4+(3×X)+(0×Y)+(0×Z)

4+(3×X)

X+Y+Z

Z+(2×Y)+(4×X)

X-Y-Z

X+Y+Z+W
```

18.2 Take each result of Exercise 18.1 and (without looking at the original expression in the exercise) write an equivalent expression in terms of the arguments *X* and *Y* (and if necessary, *Z* and *W*). Compare your results with the original expressions.

18.3 Let `X←3` and `Y←2` and `Z←4` and `W←15` and let `V←X,Y` or `V←X,Y,Z` or `V←X,Y,Z,W` as appropriate. Then evaluate each expression of Exercise 18.1 and evaluate each equivalent expression which you obtained and compare the results.

18.4 a) Determine a vector *A* and a matrix *B* such that the expression `A+B+.×X,Y` is equivalent to the following pair of expressions:

```
3+(2×X)+(¯4×Y)
4+(¯3×X)+(2×Y)
```

More precisely, `A+B+.×X,Y` is equivalent to the catenation of these expressions, that is:

```
(3+(2×X)+(¯4×Y)),4+(¯3×X)+(2×Y)
```

b) Evaluate *A*+*B*+.×*X*,*Y* and compare the result with the result of evaluating the given expressions for each of the following pairs of values of *X* and *Y*:

| *X* | *Y* |
|---|---|
| 2 | 5 |
| 3 | 0 |
| 0 | 3 |
| 0 | 0 |
| ¯4 | 2 |
| 3 | ¯7 |
| ¯9 | ¯3 |

c) Take the result of Part (a) and from it write the equivalent expressions in terms of *X* and *Y* and compare with the original expressions.

18.5 Repeat Exercise 18.4 for the following pairs of expressions:

¯3+(4×*X*)+(¯2×*Y*)
6+(2×*X*)+(7×*Y*)

¯3-(¯4×*X*)+(2×*Y*)
6-(¯2×*X*)+(¯7×*Y*)

(3×*X*)+(7×*Y*)
(4×*Y*)+(8×*X*)

2+3×*X*
8+7×*Y*

18.6 Choosing any values that you wish for *Z* in the evaluations, repeat Exercise 18.4 for the following set of expressions:

18+(3×*X*)+(¯4×*Y*)+(7×*Z*)
¯13+(2×*Y*)
2+(0×*X*)+(3×*Y*)+(¯4×*Z*)

18.7 a) Plot the mapping produced by the expression *A*+*B*+.×*V* for the following set of values:

```
  A           B          V
3 ¯5       2 ¯1        2  1
          ¯3  4
```

b) Add to the plot of Part (a) the mappings for each of the following 7 values of *V* (shown in columns to save space):

```
¯2   0   0   1   1  ¯1  ¯1.4
¯1   0   1   0   1  ¯1    .2
```

c) Make other maps for any values of *A* and *B* that you wish to choose. For each case try to find some value of *V* which (like the last one in Part (b))maps into the origin (that is, the point 0 0).

18.8 Repeat Exercise 18.7 but with *A* assigned the value 0 0 in every case.

18.9 Let *B* be the following matrix:

```
  .5   .866
¯.866  .5
```

a) Plot the mapping *B*+.×*V* when applied to each of the set of points *V* listed in exercise 18.7 (b).

b) Verify that this mapping is a rotation.

18.10 Repeat Exercise 18.9 for each of the following values of the matrix *B*:

```
 0  1      0 ¯1     ¯1  0      1  0
¯1  0      1  0      0 ¯1      0  1

 .707  .707          .707 ¯.707
¯.707  .707          .707  .707
```

18.11 a) Let $B$ be the matrix of Exercise 18.9. Then plot the mappings produced by repeated applications of $B$ to the point `V←1 2`, that is:

```
B+.×V
B+.×B+.×V
B+.×B+.×B+.×V
```

and so forth.

b) How many applications of $B$ are equivalent to the identity function?

c) Write an expression of the form `B+.×B+.×B+.×B`, with $N$ occurrences of $B$, where $N$ denotes the answer to Part (b). Evaluate this expression and compare the result with the identity matrix.

18.12 a) Repeat Exercise 18.11 for each of the matrices of Exercise 18.10.

b) Determine a rotation matrix whose first and last elements are `.5*.5` and repeat Exercise 18.11 for this matrix.

18.13 a) Let $B$ be a rotation matrix with elements $S$, $C$, $-C$, and $S$ as defined at the beginning of Section 18.3. Show that the product `B+.×⍉B` is the identity matrix.

b) Show that `(⍉B)+.×B` is the identity matrix.

c) Test these results by applying them to the rotation matrices of Exercise 18.10

18.14 Plot the mapping produced by the translation `A+V` applied to each of the points $V$ of Exercise 18.7 (b) for each of the following cases:

a) `A←3 ¯5`
b) `A←3 0`
c) `A←¯3 0`
d) `A←0 4`

18.15 Let $M$ be the matrix given for $V$ in Exercise 18.7 (b), that is, the columns of $M$ are the values of $V$ in the order shown.

a) Evaluate the expression `B+.×M`, where $B$ is the matrix of Exercise 18.9. Compare the results with those of Exercise 18.9.

b) Repeat Part (a) for the matrices $B$ listed in Exercise 18.10.

18.16 Define a matrix $P$ to be used with the matrices $B$ and $M$ of Exercise 18.15 in the expression `P+B+.×M` to produce the translation `3 ¯5`.

18.17 Use the matrices $P$ and $M$ of Exercise 18.16 and the matrix `B←2 2⍴0 1 ¯1 0` and plot the mappings produced by each of the following:

```
P+B+.×M

B+.×P+M

(B+.×P)+(B+.×M)
```

18.18 a) Define a stretching matrix $B$ and apply it to the matrix $M$ of Exercise 18.15, that is, evaluate the expression `B+.×M`.

b) Compare the matrices $M$ and `B+.×M` and state the relation between them.

c) Repeat Part (a) for a number of stretching matrices which you choose.

18.19 a) Choose a number of matrices and use them to test the distributivity of the inner product +.× over +.

b) Choose a number of matrices and use them to test the associativity of the +.× inner product.

18.20 Let *A*, *B*, and *C* be 2-by-2 matrices and give names to each of the elements according to the following scheme:

```
A11 A12     B11 B12     C11 C12
A21 A22     B21 B22     C21 C22
```

a) For each of the following expressions write an equivalent expression in terms of the names *A*11, *A*12, etc:

```
B+.×C

A+.×(B+.×C)

(A+.×B)+.×C
```

b) Prove that the expression obtained for the second case of Part (a) is equivalent to the expression obtained for the third case. (This proves the associativity of +.× for 2-by-2 matrices.)

18.21 Repeat Exercise 18.20, replacing the second and third expressions of Part (a) by the following expressions

```
A+.×(B+C)

(A+.×B)+(A+.×C)
```

(This proves that +.× distributes over + for 2-by-2 matrices.)

18.22 a) Make a 3-dimensional plot of the eight points represented by the following matrix *M*:

```
              M
1  2  3  0  0  0  ¯1  1
1  2  3  1  2  0   2 ¯1
1  2  3  1  2  0  ¯3  1
```

b) Evaluate the expression *B*+.×*M* for the following matrix *B*:

```
    B
2   0  ¯1
1  ¯2   1
1   1   1
```

c) Add to the plot the points determined in Part (b) and show the mapping produced by the matrix *B*.

18.23 a) Choose any three 3 by 3 matrices *C D* and *E* and use them to test the associativity of the +.× inner product in three dimensions.

b) Use the same matrices to test the distributivity of +.× over +.

18.24 a) Make a plot to show the mapping *B*+.×*M*, where *B* is the following 3-dimensional rotation matrix:

```
1      0      0
0    .707   .707
0   ¯.707   .707
```

and *M* is the matrix of points given in Exercise 18.22.

b) Repeat Part (a) for any 3-dimensional rotation matrices you may wish to construct.

18.25 a) Evaluate the following:

```
X←0 1 2 3 4 5 6 7 8 9 10
Y←⌽X
Y
```

```
M←(2×Y)∘.+(X-12)
M

N←Y∘.+¯1×X
N

0=M

0=N

' *'[1+0=M]

' *'[1+0=N]

(0=M)∨(0=N)

(0=M)∧(0=N)
```

b) Discuss the results of Part (a), stating as clearly as you can what each of the logical matrices represent.

c) Repeat Part (a) for various values of *X* and *Y* and for various linear functions of your own choosing.

# 19

19.1 a) Test the fact that the 2-dimensional matrices *B* and *IB* given in Section 19.2 actually produce inverse functions by applying them to the set of points represented by the following matrix *M*:

```
1  2  0  1  ¯3  ¯5  1  0
2  5  0  1   5  ¯2  0  1
```

b) Evaluate the expressions `B+.×IB` and `IB+.×B` and compare them with the identity matrix.

19.2 Repeat Exercise 19.1 for the 3-dimensional matrices *B* and *IB* given in Section 19.2 and for the following matrix *M*:

```
 9  ¯3  1  0  0  ¯8  0
16   5  0  1  0   1  0
20  ¯7  0  0  1   5  0
```

19.3 a) Evaluate the expression `∧/3 ¯7=B+.×V` for the matrix `B←2 2ρ1 ¯3 ¯2 4` and for each of the following values of the 2-element vector *V*:

```
1  0.5  4.5  ¯3.2  1  0
2  3.5  0.5   4.2  0  1
```

b) Use the results of Part (a) to determine which of the given values of *V* is a solution of the equation `3 ¯7=B+.×V`.

19.4 Let *M* and *N* be the following matrices:

```
          M
¯7  5  ¯1   1  ¯5  ¯9
 3  3   8  ¯2   0   6

               N
¯18  ¯10  10  ¯14    2  ¯2
 39    5  10   22  ¯11  41
```

Each column of *M* (that is `M[;I]`) is a solution of the equation `N[;J]=B+.×M[;I]` for some *J*th column of *N*, where *B* is the matrix `B←2 2ρ2 0 ¯1 5`. Determine which column of *M* gives the solution of the equation for each column of *N*.

19.5 If *B*←2 2⍴2 3 3 5, then the basic solutions *V*1 and *V*2 are among the columns of the following matrix:

```
1  1 ¯3  0  5 ¯1.5  2
0 ¯2  2  1 ¯3  3.5  0
```

a) Determine the basic solutions of *B*

b) Using the values of *V*1 and *V*2 obtained in Part (a), evaluate the following:

```
N←(4×0 1)+(¯2×1 0)
N

V←(4×V1)+(¯2×V2)
V

B+.×V

∧/N=B+.×V
```

c) Use the scheme suggested by Part (b) to determine a solution to the equation *N*=*B*+.×*V* for the following values of *N*:

```
 5  7
¯3  8
 0  0
¯7  0
 0  4
```

19.6 The basic solutions for the matrix *B*←2 2⍴4 2 7 3 also occur among the columns of the matrix give in Exercise 19.5. Use this fact to repeat the work of Exercise 19.5 for this value of *B*.

19.7 Let *B* be the following matrix:

```
2  3
3  5
```

a) Determine a value for *VA* such that the second element of *B*+.×*VA* is zero.

b) Determine a value of *K* such that if *V*1←*VA*÷*K*, then *V*1 is a basic solution of *B*.

19.8 The vector *VA*←0 0 would satisfy the requirement imposed in Part (a) of Exercise 19.7, namely that the second element of *B*+.×*VA* must be zero. Try to use this value of *VA* to determine a basic solution *V*1 as in Part (b) of the same exercise. Whey does it not work?

19.9 Repeat Exercise 19.7 for each of the following values of *B*:

```
4  2     2  3     8  6
7  3     2  8    ¯6  8
```

19.10 a) Repeat the steps of Exercise 19.7 but modified to determine the <u>second</u> basic solution *V*2.

b) Repeat Part (a) for the matrices of Exercise 19.9.

19.11 Determine basic solutions for each of the following matrices:

```
2 7     4  3     16  5     6 9
1 3     8 11     ¯8 10     3 5
```

19.12 a) Evaluate the determinant of each matrix of Exercise 19.11

b) Evaluate the determinant of each matrix of Exercise 19.9

19.13 a) construct a matrix *B* whose determinant is 4

b) If the determinant of *B* is 4, what is the determinant of the matrix -*B*?

c) Modify the matrix *B* of Part (a) to obtain a matrix whose determinant is ¯4

d) Construct at least 3 different matrices whose determinants have the same value 100

e) Construct at least 3 different matrices whose determinants have the value 1.

19.14 What effect does each of the following changes to a matrix have on the value of its determinant:

a) Interchanging its two rows?

b) Interchanging its columns?

c) Interchanging the rows and then interchanging the columns?

e) Changing the sign of every element?

19.15 a) Evaluate the determinant of the following matrix:

```
6  12
4   8
```

b) Is it possible to determine basic solutions for this matrix?

c) Construct at least three different matrices for which it is impossible to determine basic solutions.

19.16 Determine the matrix *BS* which gives the basic solution in matrix form for each of the following matrices:

```
3  7      8  4
1  3      5  3
```

19.17 Determine the matrix of the basic solutions for each of the matrices of Exercise 19.11 and compare the results with those of Exercise 19.11.

19.18 a) Use the results of Exercises 19.16 and 19.17 to determine the solution of the equation 3 13=*B*+.×*V* for each of the matrices *B* involved in those exercises.

19.19 Find solutions to the equation

```
∧/N=(2 2ρ7 5 5 3)+.×V
```

for the following values of *N*:

```
10  23
14  12
17   3
 1   0
 0   1
```

19.20 a) Determine *BS* as the matrix of basic solutions for the matrix *B*←2 2ρ9 4 4 2

b) Evaluate the expressions:

```
B+.×M

BS+.×B+.×M

BS+.×M

B+.×BS+.×M
```

for the matrix *M* given below:

```
1  3  1  ¯7  0  0  25
1  5  0   6  1  0   3
```

19.21 Repeat Exercise 19.20 for the values of the matrix *B*:

```
4  7     13  ¯3     12  2
8 11      3   7     11  6
```

19.22 a) For the matrices *B* and *BS* of Exercise 19.20, evaluate the following:

```
B+.×BS

BS+.×B
```

b) Repeat Part (a) for each of the pairs *B* and *BS* of Exercise 19.21.

19.23 If *BS* is the matrix of basic solutions for *B*, then `B+.×BS` is always equal to `BS+.×B` (since each is equal to the identity matrix). This might suggest that the function `+.×` is commutative. Show that this is not so by constructing at least one pair of matrices *C* and *D* such that `C+.×D` is <u>not</u> equal to `D+.×C`.

19.24 a) Use the Gauss-Jordan method to determine the matrix *BS* of basic solutions for the matrix *B* of Exercise 19.20. Show all of your work.

b) Repeat Part (a) for each of the matrices of Exercise 19.21.

19.25 a) Apply the efficient method of Section 19.13 to solving the equation

```
∧/3 ¯11=B+.×V
```

for the matrix *B* of Exercise 19.20. Show all of your work.

b) Repeat Part (a) for each of the matrices of Exercise 19.21.

19.26 a) Use the Gauss-Jordan method to determine the matrix *BS* which is inverse to the following matrix *B*:

```
4  4  0
3  2  1
2  1  0
```

carry all calculations to 4 decimal places.

b) Check your result by evaluating the expression `B+.×BS`.

c) Use the matrix *BS* to obtain the solution to the equation `∧/2 ¯5 6=B+.×V`

19.27 Repeat Exercise 19.26 for each of the following matrices:

```
¯5  2   7        12  ¯8   4
 8  1  ¯3         3  17   2
 1  4   2         1   9  16
```

19.28 Apply the efficient method of solution to solve the following equation:

```
∧/12 3 14=B+.×V
```

where *B* is the following matrix:

```
10   3  14
 2  12   1
¯4   7  15
```

19.29 Evaluate the expression `⌹B`, where *B* is the matrix of Exercise 19.28.

19.30 Define a function *F* which is equivalent to the function `⌹` when applied to a 2 by 2 matrix argument.

19.31 Define a function *G* which is equivalent to the function `⌹` when applied to a 3 by 3 matrix argument. Base the function definition on the Gauss-Jordan method and use iteration as much as possible.

19.32 Modify the definition of the function *G* of Exercise 19.31 so that it applies to a square matrix argument of any dimension.

19.33 Apply the efficient method of Section 19.13 to the 5 by 6 matrix given in Section 19.16. Compare the result with the solution *C* given in the same section.

19.34 Apply the general curve fitting process to the following function table:

| *X* | *Y* |
|---|---|
| 1 | 1 |
| 3 | 6 |
| 8 | 36 |

# A

# Algebra as a Language

## A.1 INTRODUCTION

Although few mathematicians would quarrel with the proposition that the algebraic notation taught in high school is a language (and indeed the primary language of mathematics), yet little attention has been paid to the possible implications of such a view of algebra. This paper adopts this point of view to illuminate the inconsistencies and deficiencies of conventional notation and to explore the implications of analogies between the teaching of natural languages and the teaching of algebra. Based on this analysis it presents a simple and consistent algebraic notation, illustrates its power in the exposition of some familiar topics in algebra, and proposes a basis for an introductory course in algebra. Moreover, it shows how a computer can, if desired, be used in the teaching process, since the language proposed is directly usable on a computer terminal.

## A.2 ARITHMETIC NOTATION

We will first discuss the notation of arithmetic, i.e., that part of algebraic notation which does not involve the use of variables. For example, the expressions 3-4 and (3+4)-(5+6) are arithmetic expressions, but the expressions 3-*X* and (*X*+4)-(*Y*+6) are not. We will now explore the anomalies of arithmetic notation and the modifications needed to remove them.

<u>Functions and symbols for functions</u>. The importance of introducing the concept of "function" rather early in the mathematical curriculum is now widely recognized. Nevertheless, those functions which the student encounters first are usually referred to not as "functions" but as "operators". For example, absolute value (|-3|) and arithmetic negation (-3) are usually referred to as operators. In fact, most of the functions which are so fundamental and so widely used that they have been assigned some graphic symbol are commonly called operators (particularly those functions such as plus and times which apply to two arguments), whereas the less common functions which are usually referred to by writing out their names (e.g., Sin, Cos, Factorial) are called functions.

This practice of referring to the most common and most elementary functions as operators is surely an unnecessary obstacle to the understanding of functions when that term is first applied to the more complex functions encountered. For this reason the term "function" will be used here for all functions regardless of the choice of symbols used to represent them.

The functions of elementary algebra are of two types, taking either one argument or two. Thus addition is a function of two arguments (denoted by $X+Y$) and negation is a function of one argument (denoted by $-Y$). It would seem both easy and reasonable to adopt one form for each type of function as suggested by the foregoing examples, that is, the symbol for a function of two arguments occurs between its arguments, and the symbol for a function of one argument occurs before its argument. Conventional notation displays considerable anarchy on this point:

1. Certain functions are denoted by any one of several symbols which are supposed to be synonomous but which are, however, used in subtly different ways. For example, in conventional algebra $X\times Y$ and $XY$ both denote the product of $X$ and $Y$. However, one would write either $3\times Y$ or $3X$ or $X\times 3$, or $3\times 4$, but would not likely accept $X3$ as an expression for $X\times 3$, nor $3\ 4$ as an expression for $3\times 4$. Similarly, $X\div Y$ and $X/Y$ are supposed to be synonomous, but in the sentence "Reduce $8/6$ to lowest terms", the symbol / does not stand for division.

2. The power function has no symbol, and is denoted by position only, as in $X^N$. The same notation is often used to denote the $N$th element of a family or array $X$.

3. The remainder function (that is, the integer remainder on dividing $X$ into $Y$) is used very early in arithmetic (e.g., in factoring) but is commonly not recognized as a function on a par with addition, division, etc., nor assigned a symbol. Because the remainder function has no symbol and is commonly evaluated by the method of long division, there is a tendency to confuse it with division. This confusion is compounded by the fact that the term "quotient" itself is ambiguous, sometimes meaning the quotient and sometimes the integer part of the quotient.

4. The symbol for a function of one argument sometimes occurs before the argument (as in $-4$) but may also occur after it (as in $4!$ for factorial $4$) or on both sides (as in $|X|$ for absolute value of $X$).

Table A.1 shows a set of symbols which can be used in a simple consistent manner to denote the functions mentioned thus far, as well as a few other very useful basic functions such as maximum, minimum, integer part, reciprocal, and exponential. The table shows two uses for each symbol, one to denote a <u>monadic</u> function (i.e., a function of one argument), and one to denote a <u>dyadic</u> function (i.e., a function of two arguments). This is simply a systematic exploitation of the example set by the familiar use of the minus sign, either as a dyadic function (i.e., subtraction as in 4-3) or as a monadic function (i.e., negation as in -3). No function symbol is permitted to be elided; for example, *X*×*Y* may not be written as *XY*.

| Monadic form f*B* | | f | Dyadic form *A*f*B* | |
|---|---|---|---|---|
| Definition or example | Name | | Name | Definition or example |
| +3 ↔ 0+3 | Plus | + | Plus | 2+3.2 ↔ 5.2 |
| -3 ↔ 0-3 | Negative | - | Minus | 2-3.2 ↔ ¯1.2 |
| ×3 ↔ (3>0)-(3<0) | Signum | × | Times | 2×3.2 ↔ 6.4 |
| ÷3 ↔ 1÷3 | Reciprocal | ÷ | Divide | 2÷3.2 ↔0.625 |
| *B* \| ⌈*B* \| ⌊*B*<br>3.14 \| 4 \| 3 | Ceiling | ⌈ | Maximum | 3⌈7 ↔ 7 |
| ¯3.14 \| ¯3 \| ¯4 | Floor | ⌊ | Minimum | 3⌊7 ↔ 3 |
| *3 ↔ (2.71828∘∘)*3 | Exponential | * | Power | 2*3 ↔ 8 |
| ⍟*5 ↔ 5 ↔ *⍟5 | Natural logarithm | ⍟ | Logarithm | 10⍟3↔Log 3 base 10<br>10⍟3↔(⍟3)÷⍟10 |
| \|¯3.14 ↔ 3.14 | Magnitude | \| | Remainder | 3\|8 ↔ 2 |

Table A.1

A little experimentation with the notation of Table A.1 will show that it can be used to express clearly a number of matters which are awkward or impossible to express in conventional notation. For example, *X*÷*Y* is the quotient of *X* divided by *Y*; either ⌊(*X*÷*Y*) or ((*X*-(*Y*|*X*))÷*Y* yield the integer part of the quotient of *X* divided by *Y*; and *X*⌈(-*X*) is equivalent to |*X*.

In conventional notation the symbols <, ≤, =, ≥, >, and ≠ are used to state relations among quantities; for example, the expression 3<4 asserts that 3 <u>is</u> less than 4. It is more useful to employ them as symbols for dyadic functions defined to yield the value 1 if the indicated relation actually holds, and the value zero if it does not. Thus 3≤4 yields the value 1, and 5+(3≤4) yields the value 6.

Arrays. The ability to refer to collections or arrays of items is an important element in any natural language and is equally important in mathematics. The notation of vector algebra embodies the use of arrays (vectors, matrices, 3-dimensional arrays, etc.) but in a manner which is difficult to learn and limited primarily to the treatment of linear functions. Arrays are not normally included in elementary algebra, probably because they are thought to be difficult to learn and not relevant to elementary topics.

A vector (that is, a 1-dimensional array) can be represented by a list of its elements (e.g., 1 3 5 7) and all functions can be assumed to be applied element-by-element. For example:

```
      1 2 3 4 × 4 3 2 1  produces
4 6 6 4
```

Similarly:

```
      1 2 3 4 + 4 3 2 1
5 5 5 5
      ! 1 2 3 4
1 2 6 24
      1 2 3 4 * 2
1 4 9 16
      2 * 1 2 3 4
2 4 8 16
```

In addition to applying a function to each element of an array, it is also necessary to be able to apply some specified function to the collection itself. For example, "Take the sum of all elements", or "Take the product of all elements", or "Take the maximum of all elements". This can be denoted as follows:

```
      +/2 5 3 2
12
      ×/2 5 3 2
60
      ⌈/2 5 3 2
5
```

The rules for using such vectors are simple and obvious from the foregoing examples. Vectors are relevant to elementary mathematics in a variety of ways. For example:

1. They can be used (as in the foregoing examples) to display the patterns produced by various functions when applied to certain patterns of arguments.

2. They can be used to represent points in coordinate geometry. Thus 5 7 19 and 2 3 7 represent two points, 5 7 19 - 2 3 7 yields 3 4 12, the displacement between

them, and (+/(5 7 19 - 2 3 7)*2)*.5 yields 13, the distance between them.

3. They can be used to represent rational numbers. Thus if 3 4 represents the fraction three-fourths, then 3 4×5 6 yields 15 24, the product of the fractions represented by 3 4 and 5 6. Moreover, ÷/3 4 and ÷/5 6 and ÷/15 24 yield the actual numbers represented.

4. A polynomial can be represented by its vector of coefficients and vector of exponents. For example, the polynomial with coefficients 3 1 2 4 and exponents 0 1 2 3 can be evaluated for the argument 5 by the following expression:

```
      +/3 1 2 4 × 5 * 0 1 2 3
558
```

Constants. Conventional notation provides means for writing any positive constant (e.g., 17 or 3.14) but there is no distinct notation for negative constants, since the symbol - occurring in a number like -35 is indistinguishable from the symbol for the negation function. Thus negative thirty-five is written as an expression, which is much as if we neglected to have symbols for five and zero because expressions for them could be written in a variety of ways such as 8-3 and 8-8.

It seems advisable to follow Beberman [1] in using a raised minus sign to denote negative numbers. For example:

```
      3 - 5 4 3 2 1
¯2 ¯1 0 1 2
```

Conventional notation also provides no convenient way to represent numbers which are easily expressed in expressions of the form $2.14\times10^{8}$ or $3.265\times10^{\bar{}9}$. A useful practice widely used in computer languages is to replace the symbols ×10 by the symbol *E* (for exponent) as follows: 2.14*E*8 and 3.265*E*¯9.

Order of execution. The order of execution in an algebraic expression is commonly specified by parentheses. The rules for parentheses are very simple, but the rules which apply in the absence of parentheses are complex and chaotic. They are based primarily on a hierarchy of functions (e.g., the power function is executed before multiplication, which is executed before addition) which has apparently arisen because of its convenience in writing polynomials.

Viewed as a matter of language, the only purpose of such rules is the potential economy in the use of parentheses and the consequent gain in readability of complex expressions. Economy and simplicity can be achieved by the following rule: parentheses are obeyed as usual and otherwise expressions are evaluated from right to left with all functions being treated equally. The advantages of this rule and the complexity and ambiguity of conventional rules are discussed in Berry [2], page 27 and in Iverson [3], Appendix A. Even polynomials can be conveniently written without parentheses if use is made of vectors. For example, the polynomial in $X$ with coefficients 3 1 2 4 can be written without parentheses as +/3 1 2 4 × $X$ ⋆ 0 1 2 3. Moreover, Horner's expression for the efficient evaluation of this same polynomial can also be written without parentheses as follows:

3+$X$×1+$X$×2+$X$×4

Analogies with natural language. The arithmetic expression 3×4 can be viewed as an order to do something, that is, multiply the arguments 3 and 4. Similarly, a more complex expression can be viewed as an order to perform a number of operations in a specified order. In this sense, an arithmetic expression is an imperative sentence, and a function corresponds to an imperative verb in natural language. Indeed, the word "function" derives from the latin verb "fungi" meaning "to perform".

This view of a function does not conflict with the usual mathematical definition as a specified correspondence between the elements of domain and range, but rather supplements this static view with a dynamic view of a function as that which produces the corresponding value for any specified element of the domain.

If functions correspond to imperative verbs, then their arguments (the things upon which they act) correspond to nouns. In fact, the word "argument" has (or at least had) the meaning topic, theme, or subject. Moreover, the positive integers, being the most concrete of arithmetical objects, may be said to correspond to proper nouns.

What are the roles of negative numbers, rational numbers, irrational numbers, and complex numbers? The subtraction function, introduced as an inverse to addition, yields positive integers in some cases but not in others, and negative numbers are introduced to refer to the results in these cases. In other words, a negative number refers to a process or the result of a process, and is therefore analogous to an abstract noun. For example, the abstract noun "justice" refers not to some concrete object (examples of which one may point to) but to a process or result of a process. Similarly, rational and complex numbers refer to the results of processes; division, and finding the zeros of polynomials, respectively.

## A.3 ALGEBRAIC NOTATION

Names. An expression such as $3\times X$ can be evaluated only if the variable $X$ has been assigned an actual value. In one sense, therefore, a variable corresponds to a pronoun whose referent must be made clear before any sentence including it can be fully understood. In English the referent may be made clear by an explicit statement, but is more often made clear by indirection (e.g., "See the door. Close it."), or by context.

In conventional algebra, the value assigned to a variable name is usually made clear informally by some statement such as "Let $X$ have the value 6" or "Let $X$=6". Since the equal symbol (that is, '=') is also used in other ways, it is better to avoid its use for this purpose and to use a distinct symbol as follows:

```
      X←6
      Y←3×4
      X+Y
18
      (X-3)×(X-5)
3
```

Assigning names to expressions. In the foregoing example, the expression $(X-3)\times(X-5)$ was written as an instruction to evaluate the expression for a particular value already assigned to $X$. One also writes the same expression for the quite different notion "Consider the expression $(X-3)\times(X-5)$ for any value which might later be assigned to the argument $X$." This is a distinct notion which should be represented by distinct notation. The idea is to be able to refer to the expression and this can be done by assigning a name to it. The following notation serves:

```
      ∇ Z ← G X
      Z←(X-3)×(X-5)∇
```

The ∇'s indicate that the symbols between them define a function; the first line shows that the name of the function is $G$. The names $X$ and $Z$ are dummy names standing for the argument and result, and the second line shows how they are related.

Following this definition, the name $G$ may be used as a function. For example:

```
      G 6
3
      G 1 2 3 4 5 6 7
8 3 0 ¯1 0 3 8
```

Iterative functions can be defined with equal ease as shown in Chapter 12.

Form of names. If the variables occurring in algebraic sentences are viewed simply as names, it seems reasonable to employ names with some mnemonic significance as illustrated by the following sequence:

$LENGTH \leftarrow 6$
$WIDTH \leftarrow 5$
$AREA \leftarrow LENGTH \times WIDTH$
$HEIGHT \leftarrow 4$
$VOLUME \leftarrow AREA \times HEIGHT$

This is not done in conventional notation, apparently because it is ruled out by the convention that the multiplication sign may be elided; that is, *AREA* cannot be used as a name because it would be interpreted as $A \times R \times E \times A$.

This same convention leads to other anomalies as well, some of which were discussed in the section on arithmetic notation. The proposal made there (i.e., that the multiplication sign cannot be elided) will permit variable names of any length.

## A.4 ANALOGIES WITH THE TEACHING OF NATURAL LANGUAGE

If one views the teaching of algebra as the teaching of a language, it appears remarkable how little attention is given to the reading and writing of algebraic sentences, and how much attention is given to identities, that is, to the analysis of sentences with a view to determining other equivalent sentences; e.g., "Simplify the expression $(X-4) \times (X+4)$." It is possible that this emphasis accounts for much of the difficulty in teaching algebra, and that the teaching and learning processes in natural languages may suggest a more effective approach.

In the learning of a native language one can distinguish the following major phases:

1. An informal phase, in which the child learns to communicate in a combination of gestures, single words, etc., but with no attempt to form grammatical sentences.

2. A formal phase, in which the child learns to communicate in formal sentences. This phase is essential because it is difficult or impossible to communicate complex matters with precision without imposing some formal structure on the language.

3. An analytic phase, in which one learns to analyze sentences with a view to determining equivalent (and perhaps "simpler" or "more effective") sentences. The extreme case of such analysis is Aristotelian Logic, which attempts a formal analysis of certain classes of sentences. More practical everyday cases occur every

time one carefully reads a composition and suggests alternative sentences which convey the same meaning in a briefer or simpler form.

The same phases can be distinguished in the teaching of algebraic notation:

1. An informal phase in which one issues an instruction to add 2 and 3 in any way which will be understood. For example:

   2+3 Add 2 and 3

   $$\begin{array}{r}2\\ 3\\ \hline\end{array} \qquad \begin{array}{r}2\\ +3\\ \hline\end{array}$$

   Add two and three

   Add // and ///

   The form of the expression is unimportant, provided that the instruction is understood.

2. A formal phase in which one emphasizes proper sentence structure and would not accept expressions such as $6 \times \begin{array}{r}2\\ \underline{3}\end{array}$ or 6×(add two and three) in lieu of 6×(2+3). Again, adherence to certain structural rules is necessary to permit the precise communication of complex matters.

3. An analytic phase in which one learns to analyze sentences with a view to establishing certain relations (usually identity) among them. Thus one learns not only that 3+4 is equal to 4+3 but that the sentences $X+Y$ and $Y+X$ are equivalent, that is, yield the same result whatever the meanings assigned to the pronouns $X$ and $Y$.

In learning a native language, a child spends many years in the informal and formal phases (both in and out of school) before facing the analytic phase. By this time she has easy familiarity with the purposes of a language and the meanings of sentences which might be analyzed and transformed. The situation is quite different in most conventional courses in algebra - very little time is spent in the formal phase (reading, writing and "understanding" formal algebraic sentences) before attacking identities (such as commutativity, associativity, distributivity, etc.). Indeed, students often do not realize that they might quickly check their work in "simplification" by substituting certain values for the variables occurring in the original and derived expressions and comparing the evaluated results to see if the expressions have the same "meaning", at least for the chosen values of the variables.

It is interesting to speculate on what would happen if a native language were taught in an analogous way, that is, if children were forced to analyze sentences at a stage in their development when their grasp of the purpose and meaning of sentences were as shaky as the algebra student's grasp of the purpose and meaning of algebraic sentences. Perhaps they would fail to learn to converse, just as many students fail to learn the much simpler task of reading.

Another interesting aspect of learning the non-analytic aspects of a native language is that much (if not most) of the motivation comes not from an interest in language, but from the intrinsic interest of the material (in children's stories, everyday dialogue, etc.) for which it is used. It is doubtful that the same is true in algebra - ruling out statements of an analytic nature (identities, etc.), how many "interesting" algebraic sentences does a student encounter?

The use of arrays can open up the possibility of much more interesting algebraic sentences. This can apply both to sentences to be read (that is, evaluated) and written by students. For example, the statements:

```
2*1 2 3 4 5
2×1 2 3 4 5
2÷1 2 3 4 5
1 2 3 4 5÷2
1 2 3 4 5*2
1 2 3 4 5×5 4 3 2 1
```

produce interesting patterns and therefore have more intrinsic interest than similar expressions involving only single quantities. For example, the last expression can be construed as yielding a set of possible areas for a rectangle having a fixed perimeter of 12.

More interesting possibilities are opened up by certain simple extensions of the use of arrays. One example of such extensions will be treated here. This extension allows one to apply any dyadic function to two vectors *A* and *B* so as to obtain not simply the element-by-element product produced by the expression *A*×*B*, but a table of all products produced by pairing each element of *A* with each element of *B*. For example:

```
A←1 2 3
B←2 3 5 7
```

```
     A∘.×B                    A∘.+B                    A∘.*B

 2   3   5   7            3   4   6   8           1    1    1    1
 4   6  10  14            4   5   7   9           4    8   32  128
 6   9  15  21            5   6   8  10           9   27  243 2187
```

If S←1 2 3 4 5 6 7, then the following expressions yield an addition table, a multiplication table, a subtraction table, a maximum table, an "equal" table, and a "greater than or equal" table:

```
     S∘.+S                                     S∘.⌈S
 2   3   4   5   6   7   8                  1  2  3  4  5  6  7
 3   4   5   6   7   8   9                  2  2  3  4  5  6  7
 4   5   6   7   8   9  10                  3  3  3  4  5  6  7
 5   6   7   8   9  10  11                  4  4  4  4  5  6  7
 6   7   8   9  10  11  12                  5  5  5  5  5  6  7
 7   8   9  10  11  12  13                  6  6  6  6  6  6  7
 8   9  10  11  12  13  14                  7  7  7  7  7  7  7

     S∘.×X                                     S∘.=S
 1   2   3   4   5   6   7                  1 0 0 0 0 0 0
 2   4   6   8  10  12  14                  0 1 0 0 0 0 0
 3   6   9  12  15  18  21                  0 0 1 0 0 0 0
 4   8  12  16  20  24  28                  0 0 0 1 0 0 0
 5  10  15  20  25  30  35                  0 0 0 0 1 0 0
 6  12  18  24  30  36  42                  0 0 0 0 0 1 0
 7  14  21  28  35  42  49                  0 0 0 0 0 0 1

     S∘.-S                                     S∘.≥S
 0 ¯1 ¯2 ¯3 ¯4 ¯5 ¯6                        1 0 0 0 0 0 0
 1  0 ¯1 ¯2 ¯3 ¯4 ¯5                        1 1 0 0 0 0 0
 2  1  0 ¯1 ¯2 ¯3 ¯4                        1 1 1 0 0 0 0
 3  2  1  0 ¯1 ¯2 ¯3                        1 1 1 1 0 0 0
 4  3  2  1  0 ¯1 ¯2                        1 1 1 1 1 0 0
 5  4  3  2  1  0 ¯1                        1 1 1 1 1 1 0
 6  5  4  3  2  1  0                        1 1 1 1 1 1 1
```

Moreover, the graph of a function can be produced as an "equal" table as follows. First recall the function *G* defined earlier:

```
     ∇Z←G X
      Z←(X-3)×(X-5)∇

     G S
8  3  0 ¯1  0  3  8
```

The range of the function for this set of arguments is from 8 down to ¯1, and the elements of this range are all contained in the following vector:

```
     R←8 7 6 5 4 3 2 1 0 ¯1
```

Consequently, the "equal" table *R*∘.=*G S* produces a rough graph of the function (represented by 1's) as follows:

```
     R∘.=G S
1 0 0 0 0 0 1
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 1 0 0 0 1 0
0 0 0 0 0 0 0
0 0 0 0 0 0 0
0 0 1 0 1 0 0
0 0 0 1 0 0 0
```

## A.5 A PROGRAM FOR ELEMENTARY ALGEBRA

The foregoing analysis suggests the development of an algebra curriculum with the following characteristics:

1. The notation used is unambigious, with simple and consistent rules of syntax, and with provision for the simple and direct use of arrays. Moreover, the notation is not taught as a separate matter, but is introduced as needed in conjunction with the concepts represented.

2. Heavy use is made of arrays to display mathematical properties of functions in terms of patterns observed in vectors and matrices (tables), and to make possible the reading, writing, and evaluation of a host of interesting algebraic sentences before approaching the analysis of sentences and the concomitant development of identities.

Such an approach has been adopted in the present text, where it has been carried through as far as the treatment of polynomials and of linear functions and linear equations. The extension to further work in polynomials, to slopes and derivatives, and to the circular and hyperbolic functions is carried forward in Chapters 4-8 of Iverson [3].

It must be emphasized that the proposed notation, though simple, is not limited in application to elementary algebra. A glance at the bibliography of Rault and Demars[4] will give some idea of the wide range of applicability.

The role of the computer. Because the proposed notation is simple and systematic it can be executed by automatic computers and has been made available on a number of time-shared terminal systems. The most widely used of these is described in Falkoff and Iverson [5]. It is important to note that the notation is executed directly, and the user need learn nothing about the computer itself. In fact, each of the examples in this appendix are shown exactly as they would be typed on a computer terminal keyboard.

The computer can obviously be useful in cases where a good deal of tedious computation is required, but it can be useful in other ways as well. For example, it can be used by a student to explore the behavior of functions and discover their properties. To do this a student will simply enter expressions which apply the functions to various arguments. If the terminal is equipped with a display device, then such exploration can even be done collectively by an entire class. This and other ways of using the computer are discussed in Berry et al [6] and in Appendix C.

# B

# The Mechanics of Computer Use

## B.1 INTRODUCTION

An APL computer consists of a typewriter connected by a telephone or wire to a remote computer. The user controls the computer only by typing on the typewriter keyboard shown in Fig. B.1. Each entry is concluded by a carriage return, that is, the computer responds only after the carriage return button is depressed. For example:

```
      3×5
15
      3.2+4.9
8.1
```

A two-position <u>mode</u> switch marked LOCAL and COMMUNICATE can be switched to the position LOCAL to make the typewriter usable as an ordinary typewriter. This mode is convenient for gaining familiarity with the keyboard.

The material in this Appendix applies almost exactly to any APL computer. Particular APL computers may, however, differ in minor details and information on such details can be found in the manual for the particular computer. These manuals also present other aspects of the APL language not treated in this text and should eventually be consulted by any serious user.

## B.2 GETTING STARTED

There are several different procedures for connecting the typewriter to the remote computer. In the case of a wired connection one simply turns the typewriter power switch to ON and the mode switch to COMMUNICATE. In the case of a phone connection one must also dial the number of the computer, wait until the computer answers with a high-pitched tone, depress the button marked DATA on the special telephone used, and then cradle the handset.

When the connection to the remote computer has been established, a user may <u>sign on</u> by typing a right parenthesis followed by the account number assigned to him. For example:

```
      )1181
001) 15.00.13 01/21/72 KEIVERSON
```

The last line above shows a typical response of the computer to a sign-on.

## B.3 USING APL

Once signed on, a user may enter any sequence of valid APL expressions and expect each to be executed as soon as the carriage return is depressed. Examples of valid APL expressions and the expected results may be found in abundance in the text.

Numbers can be entered (and may be printed by the computer) in either the familiar decimal form or in the <u>exponential</u> form presented in Section 5.17. For example:

```
      2×14.3E¯3
0.0286
      23×10000×10000×10000
2.3E13
```

Any entry can be revised at any time before pressing the carriage return by backspacing to the position of the change, striking the <u>attention</u> button, (which spaces the paper up and marks the position with a caret) and then continuing typing. For example:

```
      X←'ABCDFGH
             ∨
             EFGHIJK'
      X
ABCDEFGHIJK

      Y←'ABCDFGHIJ
             ∨
             EFG'
      Y
ABCDEFG
```

The attention button can also be used to stop the printing of any output and to interrupt the execution of any function.

## B.4 ERROR REPORTS

The entry of any invalid expression will invoke an error report. For example:

```
      128÷8-4×2
DOMAIN ERROR
      128÷8-4×2
          ∧
```

The report indicates the type of error committed, repeats the entire expression, and marks the point of difficulty. Following an error report one may continue to enter expressions in the normal way, usually revising the offending expression to the desired form. For example:

```
      128÷(8-4)×2
16
```

The various types of errors are listed in Table B.2 together with the probable cause and suggested corrective action.

A name already used for a function cannot be used as a name for a variable. For example:

```
      ∇Z←A X
[1]   Z←X+1∇

      A←17
SYNTAX ERROR
      A←17
      ∧
```

Conversely:

```
      F←3
      ∇Z←F X
DEFN ERROR
      ∇Z←F X
           ∧
```

Any name in use can, however, be freed for a different use by erasing it as follows:

```
      )ERASE A
      A←17
      )ERASE F
      ∇Z←F X
[1]   Z←X*2∇
```

If an error is detected in some line of a defined function, the error report includes (at the left) an indication of the function and the line number. Before revising or re-using the function one should escape from (i.e., terminate) it by entering a right arrow alone (that is, →).

## B.5 REVISING A FUNCTION DEFINITION

When a function is being defined, the computer helpfully prints the number of each line (in brackets) before unlocking the keyboard to permit entry of the body of the line. However, one may change the line number by beginning the entry with any desired line number enclosed in brackets. In this way one may give a revised specification to any line already entered. For example:

```
       ∇Z←G N
[1]    S←⍳N
[2]    Z←(~S∊S∘.×S)/S
[3]    [1]S←1+⍳N-1
[2]    ∇
```

The final expression on line 1 then becomes S←1+⍳N-1. The header is referred to as line 0.

The definition of a function *G* may be re-opened by entering ∇*G*. The function definition may then be revised or displayed before re-closing the definition. Display is requested by entering [⎕]. For example:

```
       ∇G
[3]    [⎕]
       ∇ Z←G N
[1]    S←1+⍳N-1
[2]    Z←(~S∊S∘.×S)/S
       ∇
[3]    ∇
```

A new line may be inserted between lines 1 and 2 by referring to it by some fraction (say [1.5]) between 1 and 2. When the function definition is closed, integer line numbers are reassigned in the obvious order. For example:

```
       ∇G
[3]    [1.5]S←2×S
[1.6]  ∇
       ∇G[⎕]∇
     ∇ Z←G N
[1]    S←1+⍳N-1
[2]    S←2×S
[3]    Z←(~S∊S∘.×S)/S
     ∇
```

## B.6 THE ACTIVE WORKSPACE

The remote computer keeps all record of the functions defined and the variables specified in a section of memory referred to as the active workspace. This workspace is limited in size (usually accomodating a few thousand numbers) and certain expressions may fail of execution for lack of space. For example, the following expression asks for the production of a huge table but would instead evoke the response shown:

```
      (⍳8000)∘.+⍳25
WS FULL
      (⍳8000)∘.+⍳25
      ∧
```

During any session, the current value of any variable and the current definition of any function remain accessible in the active workspace at all times until removal by erasure or until the entire workspace is cleared by the entry )*CLEAR*.

## B.7 TERMINATING A WORK SESSION

A work session may be terminated as follows:

```
      )CONTINUE
  15.11.50 01/21/72 CONTINUE
001   15.11.51 01/21/72 KEI
CONNECTED     0.11.38  TO DATE    68.30.53
CPU TIME      0.00.00  TO DATE     0.06.47
```

The last two lines above show a typical response by the computer giving both the length of the sesson (*CONNECTED* time) and the remote computer time used in hours, minutes, and seconds.

The next time this same account number is signed on, the work may be continued exactly as if there had been no interruption; the active workspace remains intact from the previous session.

A work session may also be terminated by the entry )*OFF* in which case the active workspace is lost and the session at the next sign-on begins with a clear workspace.

## B.8 USE OF LIBRARIES

Each account number has associated with it a library, and a copy of the active workspace can be preserved in this library under any desired name by entering )*SAVE* followed by the name. For example:

```
      )SAVE CHAPTER3
  15.12.46  01/21/72
```

Moreover, the current active workspace could then be recalled at any later time as follows:

```
      )LOAD CHAPTER3
SAVED 15.12.46  01/21/72
```

A workspace can be loaded repeatedly; to remove it from the library one must enter:

```
      )DROP CHAPTER3
```

The content of any library workspace can be added to the active workspace by entering )*COPY* followed by the workspace name. For example, the workspaces *CHAPTER*1 and *NEW* could be merged in the active workspace as follows:

```
      )LOAD CHAPTER1
SAVED 15.13.55 01/21/72
      )COPY NEW
SAVED 15.14.01 01/21/72
```

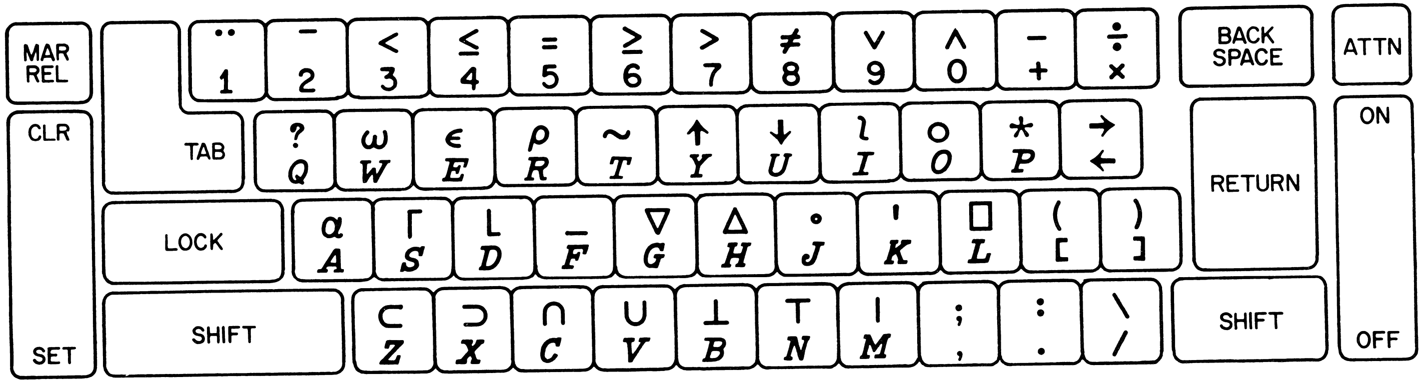
Moreover, any single function or variable can be copied from a library workspace by appending its name after the workspace as follows:

```
      )COPY CHAPTER5 X
```

The foregoing expressions concern only the user's personal library associated with his account number. He may also load or copy from any other library for which he knows the account number and workspace names; the account number is simply inserted after the words *LOAD* or *COPY*. For example:

```
      )LOAD 2073 ALGEBRA
SAVED 15.15.01 01/21/72
```

It is, of course, impossible for a user to *SAVE* into any library but his own.




| Composite Symbols | Formed by |
|---|---|
| ⌽ | \| backspace ○ |
| ⍉ | \ backspace ○ |
| ⊖ | - backspace ○ |
| ⍲ | ∧ backspace ~ |
| ⍱ | ∨ backspace ~ |
| ⍒ | ∇ backspace ~ |
| ⌹ | ÷ backspace ⎕ |

APL\360 Keyboard

Figure B.1

| TYPE | Cause; CORRECTIVE ACTION |
|---|---|
| *CHARACTER* | Illegitimate overstrike. |
| *DOMAIN* | Arguments not in the domain of the function. |
| *DEFN* | Misuse of ∇ or ⎕ symbols:<br>1. ∇ is in some position other than the first.<br>2. The function is in use. ENTER → REPEATEDLY UNTIL LIST OF FUNCTIONS IN USE (OBTAINED BY ENTERING )*SI*) IS EMPTY.<br>3. Use of other than the function name alone in reopening a definition.<br>4. Improper request for a line edit or display. |
| *INDEX* | Index value out of range. |
| *LENGTH* | Shapes not comformable. |
| *RANK* | Ranks not conformable. |
| *RESEND* | Transmission failure. RE-ENTER. IF CHRONIC, REDIAL OR HAVE TERMINAL OR PHONE REPAIRED. |
| *SYNTAX* | Invalid syntax; e.g., two variables juxtaposed; function used without appropriate arguments as dictated by its header; unmatched parentheses. |
| *VALUE* | Use of name which has not been assigned a value. ASSIGN A VALUE TO THE VARIABLE, OR DEFINE THE FUNCTION. |
| *WS FULL* | Workspace is filled (perhaps by temporary values produced in evaluating a compound expression). ENTER →, ERASE NEEDLESS OBJECTS, OR REVISE CALCULATIONS TO USE LESS SPACE. |

Error Reports

TABLE B.2

# C

# Use of the Computer in Teaching

## C.1 INTRODUCTION

The computer is an important tool of mathematics. Nevertheless, it is essential that it be treated as a <u>tool</u>, and that the details of its use not be permitted to dominate or obscure the treatment of mathematical topics. For example, it is important in algebra to introduce matters such as iteration and function definition as fundamental mathematical notions and not as ingenious tricks for making use of a computer. In order to clearly maintain this subordinate role of the computer, no mention is made of it in the body of the text, and the discussion of techniques for the use of the computer in teaching is confined to this appendix. Teachers and students may also find Iverson [7] helpful in learning to use the computer.

The discussion covers four major types of use: for experimentation, for checking solutions to exercises, as a computational tool, and for administering drill. Within these types three modes of use by the student are discussed: direct personal use, collective (e.g., classroom) use, and remote use.

## C.2 EXPERIMENTATION

The APL computer is, in effect, a mathematical laboratory; it can therefore be used to explore the properties of various mathematical functions by experimentation in their use. Such experimentation must, of course, be guided in some degree if it is to be effective. Many ideas for experimentation should flow from the exposition in the text, but the primary guidance must come from the exercises assigned. Exercises should, in fact, be designed and selected with the thought of providing guided experiments.

For example, if students have ready access to a computer the teacher might <u>not</u> explain the meanings of the various symbols ⌈, ⌊, *, |, <, ≤, etc., but rather explain the general form of expressions involving dyadic functions, exemplifying the form by familiar functions as follows:

```
      3+4
7
      3×4
12
      3÷4
0.75
```

and then ask the students to discover the meanings of the other symbols by analogous experiments on the computer.

One important general notion which can arise from such an exercise (and which does not seem to be known to present high school students) is the idea of systematic experimentation. Students seem to begin their experiments with large numbers ("Wow, it can multiply 1372 by 24967 in a flash") which give no insight. They then use smaller numbers but discover only gradually the importance of systematic experimentation. For example, execution of the sequence of expressions 3⋆4 and 5⋆2 and 6⋆7 would not give such useful clues to the properties of the function ⋆ as would the sequence 3⋆1 and 3⋆2 and 3⋆3. Moreover, students must be led to appreciate the use of arrays to organize such experiments, as in the use of expressions such as 3⋆1 2 3 and 1 2 3 4∘.⌈1 2 3 4.

Experimentation on a bare APL computer is limited to the primitive functions denoted by symbols such as ⌈, ⌊, ⋆, etc. However, one can add to this basic environment by defining any number of new functions and allowing experimentation with them. These functions may be drawn from the algebra text, although they could also be functions which describe phenomena from disciplines such as physics; the computer then becomes in some sense a physics laboratory. See, for example, the paper by Berry, et al., [8].

A student experimenting with a defined function could display and examine its definition as well as apply it to arguments. This may be desirable, but it may also be desirable to prevent such examination of the definition. This can be done by locking the function as follows: the function definition is closed not by a ∇ but by a ⍫ (formed by ∇ backspace ~). A locked function cannot be further revised or displayed. The author of a function to be locked should perhaps keep an unlocked version in some library workspace.

A student who claims to understand a function as a result of experimenting with it should be able to communicate that understanding in some way. He might for example:

1. Give the value of the result for any arguments presented to him.

2. Give a verbal statement of the function.

3. Define an equivalent function.

Any one of these three forms may be suitable for some purpose since they reflect three different levels of understanding - e.g., anyone who can give a verbal statement can also evaluate the function, but the converse is not necessarily true. The third form (i.e., the design of a suitable function definition) is not an easy task and should be first approached in terms of functions for which the first two levels of understanding have already been well assimilated.

A group of students can often be stimulated to further experimentation by the game "Guess My Function" in which any student may define (and probably lock) a function and challenge others to expose it. The exposure required may be the definition of an equivalent function, or it may be only a verbal statement or the ability to evaluate the function.

## C.3 CHECKING SOLUTIONS TO EXERCISES

The computer is convenient for checking manual solutions; one need only type in the expression to be evaluated to obtain the correct answer. Answers to questions of the type in Exercise 1.3 (which require an answer to be inserted in some expression so that it will yield a specified result) can be checked with equal ease.

There are certain advantages to allowing the student to use the computer to check his own exercises. First, this reinforces the important notion that one should always seek some method of validating a result. Second, it allows students to gain familiarity with the keyboard and other aspects of the computer in the context of a simple routine task.

## C.4 USE AS A COMPUTATIONAL TOOL

In the practical world computers are used primarily as a computational tool, applying certain defined functions (such as payroll calculation) to thousands or millions of different arguments. In teaching, this type of use is secondary since the main point is the understanding of a function rather than its application. Nevertheless, routine calculations do play some role in teaching.

For example, the functions *POLY* and *P* of Sections 13.5 and 13.6 may first be of interest for the insight they give into the meaning of the polynomial function. However, in a later study of graphing one might wish to use these known functions to evaluate points on various curves to be plotted. In this case the computer is serving only as a computational tool, allowing the accurate treatment of numerous examples which would be tedious to undertake without it.

In a series of exercises intended primarily to familiarize a student with various functions, he should often be permitted to use his own discretion in omitting many of them. On the other hand, such a series of exercises can be designed so that their results exhibit some pattern or identity which the alert student should detect, but which he will probably miss if parts of the sequence are omitted. For this reason a student may wish to complete the sequence with a minimum of tedium by using the computer to evaluate some of the exercises.

The following sequence (taken from Exercise 3.1 and shown together with the computed results) illustrates such a case:

```
      2×+/⍳5
30
      5×6
30
      2×+/⍳6
42
      6×7
42
      2×+/⍳7
56
      7×8
56
```

The foregoing results suggest that the sum of the first $N$ integers is equal to one half of the product $N \times N+1$. Moreover, the ensuing sequence in Exercise 3.1 suggests a proof of the identity.

Although most of the functions employed for computation may themselves be objects of study, there are many useful functions (called utility functions) which one may not care to understand in detail. For example, the expressions for plotting functions developed in Sections 10.11 and 10.12 are simple and worthy of study, but for practical use one may prefer a more complex plotting function which performs any necessary scaling and adds coordinate lines and labels, etc. The details of such a complex function are of interest to some people, but are of secondary interest in the study of algebra and it would therefore be viewed as a utility function.

Certain utility functions are available on all APL computer systems and are described in the corresponding manuals. There is in addition a wealth of utility functions available from users who have written their own and are pleased to exchange or share them with others via the workspace libraries provided.

## C.5 DRILL

The use of a computer to administer drill is commonly referred to by the too-inclusive term Computer Aided Instruction or CAI. Functions to administer such drill can be conveniently written in APL. For example:

```
      ∇TIMESDRILL N
[1]    Y←?N
[2]    Y
[3]    →4-3×⎕=×/Y
[4]    'WRONG, TRY AGAIN'
[5]    →2∇

      TIMESDRILL 5 30
1  23
⎕:
      23
3  16
⎕:
      38
WRONG, TRY AGAIN
3  16
⎕:
      48
2  2
⎕:
      →
```

The definition of the foregoing function employs two symbols (⎕ and ?) for functions which have not been previously defined. The occurrence of quad (⎕) causes the symbols ⎕: to print, and unlocks the keyboard to permit an entry, then evaluates that entry and substitutes the value for the ⎕ in the expression in which it occurs. In the present case the entry is compared to the product over the vector *Y* which was printed out on line 2. The roll (?) applied to a scalar *S* produces an integer chosen at random from the set ⍳*S*. When applied to a vector it applies to each of the elements in the usual manner.

Because the expressions in the definition of *TIMESDRILL* apply to vectors of any size, the function may be used for drill on the products of more than two factors as follows:

```
      TIMESDRILL 10 10 10
7  7  10
⎕:
      490
4  6  9
⎕:
      →
```

Moreover, a function employing *TIMESDRILL* can easily be written to specialize drill for any particular student. For example:

```
      ∇JOHN
[1]   TIMESDRILL 12 12∇

      JOHN
7  9
⎕:
      63
1  5
⎕:
      →
```

Drill programs of great complexity can be defined to analyze student response in great detail and to respond accordingly. Moreover, they can be designed to apply to non-mathematical topics. Examples may be found in Reference 9.

## C.6 COLLECTIVE USE

Private use of the computer by a student can provide an excellent opportunity for individual exploration. Economic considerations may, however, severely limit this kind of use and it is important to exploit collective uses of the computer. Moreover, there are some advantages of collective use which cannot be otherwise realized.

A teacher will find the computer a valuable aid for working out examples for use in class. The printed result of a session at the computer can be reproduced for distribution or be used to produce a transparency for projection in class.

Active interaction of a class with the computer can be achieved by displaying the printed results as they are produced. This can be done with a closed circuit TV or other optical projection system. With such an arrangement a teacher at the typewriter can even mediate collective experimentation by the class. This collective experimentation can be most rewarding in itself and can also suggest techniques of exploration for private use.

## C.7 REMOTE USE

The computer can be operated by a typist who simply enters expressions from sheets written and submitted by the ultimate users. The typewritten results can then be returned to each user, thus affording him remote access to the computer. This scheme has the disadvantage that the user cannot make immediate changes based on the results produced by the computer. It has the advantage that a trained typist may type much faster than the ordinary user and so make more efficient use of the computer. It also permits correspondence students and others at remote locations to use the computer.

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# Index

| | NAME | SYMBOL | DEFINITION OR EXAMPLE | SECTION # |
|---|---|---|---|---|
| D | Addition | + | 3+4←→7 | 1.2 |
| Y | Multiplication | × | 3×4←→12 | 1.2 |
| A | Subtraction | - | 3-4←→¯1 | 3.1 |
| D | Division | ÷ | 3÷4←→.75 | 5.1 |
| I | Maximum | ⌈ | 3⌈4←→4 | 2.4 |
| C | Minimum | ⌊ | 3⌊4←→3 | 2.4 |
| | Power | * | 3*4←→81   *A***B*←→×/*B*⍴*A* | 2.5 6.5-6 |
| F | Remainder | \| | 3\|4←→1 | 7.1 |
| U | Relations | <≤=≥>≠ | 3<4←→1   4<3←→0 | 4.8 |
| N | Or | ∨ | | 14.2 |
| C | And | ∧ | ∨\|0 1   ∧\|0 1   ⍱\|0 1   ⍲\|0 1 | 14.2 |
| T | Not-or | ⍱ | 0\|0 1   0\|0 0   0\|1 0   0\|1 1 | 14.2 |
| I | Not-and | ⍲ | 1\|1 1   1\|0 1   1\|0 0   1\|1 0 | 14.2 |
| O | Domino | ⌹ | *B*⌹*M* is soln of *B*=*M*+.×*X* | 19.15 |
| N | Index-of | ⍳ | 2 3 5 7⍳5 2←→3 1 | 16.1 |
| S | Repetition | ⍴ | 3⍴5←→5 5 5 | 1.7 13.3 |
| | Catenation | , | 4 2,1 3 5←→4 2 1 3 5 | 6.2 |
| | Take | ↑ | 2↑4 5 6←→4 5 | 10.5 |
| | Drop | ↓ | 2↓4 5 6←→6 | 10.5 |
| | Compression | / | 0 1 1 0/1 2 3 4←→2 3 | 7.5 |
| M | Negation | - | -4←→¯4 | 8.2 |
| O | Reciprocal | ÷ | ÷4←→.25 | 8.3 |
| N | Magnitude | \| | \|¯4←→4 | 8.4 |
| A | Factorial | ! | !4←→1×2×3×4 | 8.1 |
| D | Ceiling | ⌈ | ⌈3.4←→4 | 8.5 |
| I | Floor | ⌊ | ⌊3.4←→3 | 8.5 |
| C | Complement | ~ | ~1←→0   ~0←→1 | 8.6 |
| | Matrix Inverse | ⌹ | *M*+.×⌹*M* is the identity | 19.15 |
| | Ravel | , | | 8.7 |
| | Integers | ⍳ | ⍳4←→1 2 3 4 | 1.5 |
| | Size | ⍴ | ⍴4 1 3 6 2←→5 | 8.8 |
| | Flipping | ⌽ ⊖ ⍉ | Flip table about axis | 4.3 |
| O | Assignment | ← | *X*←6 | 1.3 |
| T | Indexing | *X*[*I*] | 2 3 5 7[2 4]←→3 7 | 4.4 |
| H | | *M*[*I*;*J*] | | |
| E | Function | ∇*Z*←*F* *X* | | 9.1-2 |
| R | Definition | ∇*Z*←*X* *F* *Y* | | |
| | Parentheses | | | 1.2 |
| | Execution order | | 3×4+5-7←→3×(4+(5-7)) | 1.2 |
| | Vectors | | 2 3 5×1 2 3←→2 6 15 | 1.6 |
| | Tables, Matrices | | | 2.1 13.3 |
| | Reduction (Over) | f/ | +/2 3 5←→10   ×/3 4←→12 | 1.4 4.10 |
| | Outer Product | ∘.f | | 2.3 |
| | Inner Product | f.g | | 13.2 13.4 |

SUMMARY OF NOTATION

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